Cuaderno de notas de trabajo

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Cuaderno 21

Órbitas del tipo elíptico

Punto de inflexión.

$$N = \frac{\ln^2 w}{w} \qquad \lim_{w\to\infty} N = \lim_{w\to\infty} \frac{2\ln w}{w} = \lim_{w\to\infty} \frac{2}{w} = 0$$

$$\frac{dN}{dw} = -\frac{\ln^2 w}{w^2} + \frac{2\ln w}{w^2} \qquad \lim_{w\to\infty} N = 0$$

$$\frac{d^2N}{dw^2} = -\frac{2\ln w}{w^3} + \frac{2\ln^2 w}{w^3} + \frac{2}{w^3} - \frac{4\ln w}{w^3}$$

$$\frac{d^2N}{dw^2} = \frac{2\ln^2 w - 6\ln w + 2}{w^3}$$

$$\ln^2 w - 3\ln w + 1 = 0$$

$$\ln w = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\ln w = \frac{3 \pm \sqrt{5}}{2} \qquad \sqrt{5} = 2.23607$$
$$3 - \sqrt{5} = 0.76393$$

$$\ln w_2 = 0.38\,97$$

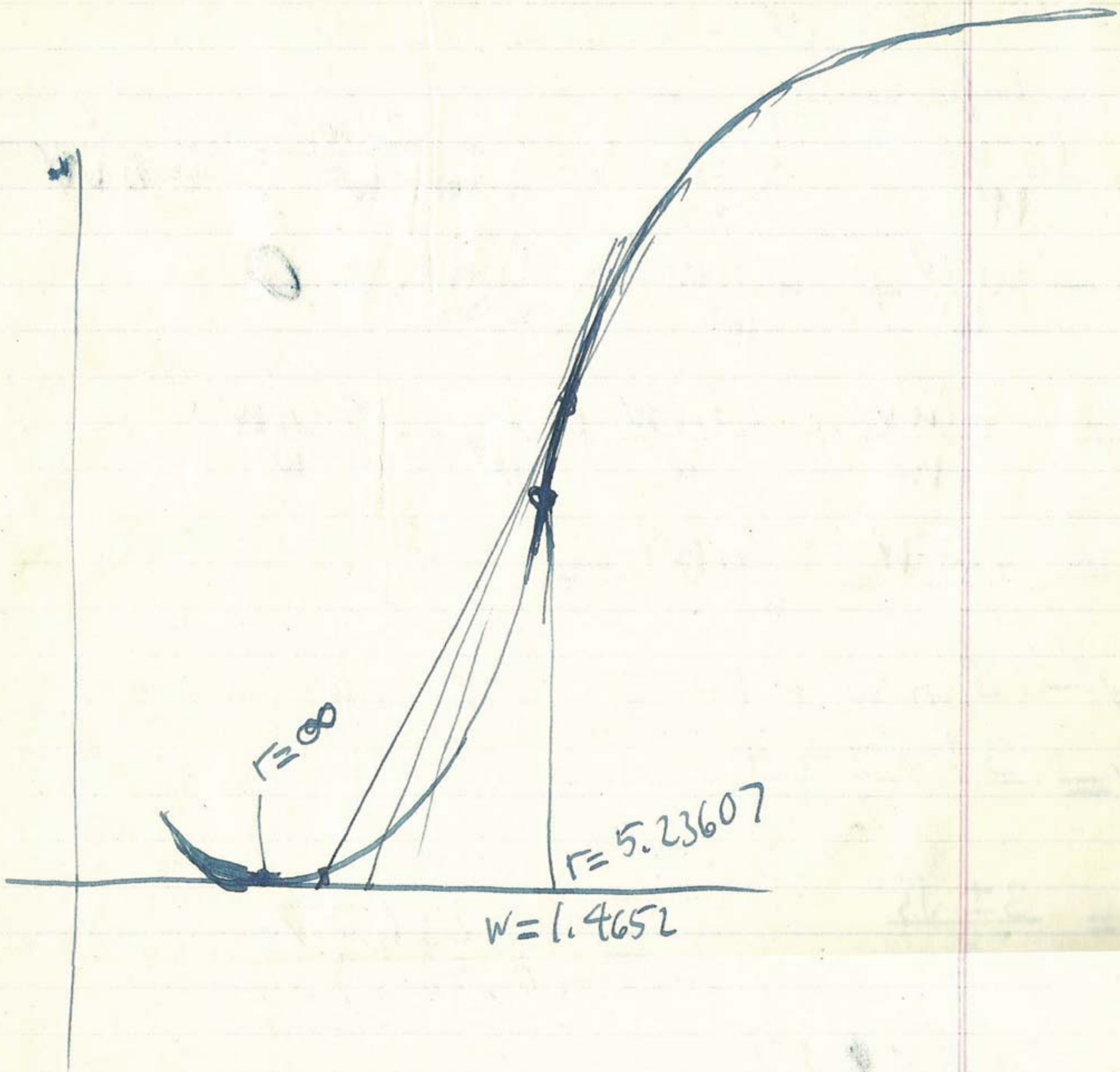
$$\log w_2 = 0.16589$$

$$\sqrt{5} = 2.236068$$
$$3 - \sqrt{5} = 0.763932$$

$$\log w_2 = 0.1658855$$

$$\ln w_2 = 0.381966$$

```
      0.381966      6
      0.434294      8
     ---------
      1527864       6
     3437694        0
     763932         3
   1527864
   1145898
  1527864
 ------------------
0.165885542004      3
```


r=∞
r=5.23607
w=1.4652

$\log N = 2\log(\ln v_2) - \log v_2$ $\qquad$ $\ln v_2 = 0.381966$

$\log(\ln v_2) = 9.58202$

$2\log(\ln v_2) = 9.16404$

$\log v_2 = 0.16589$

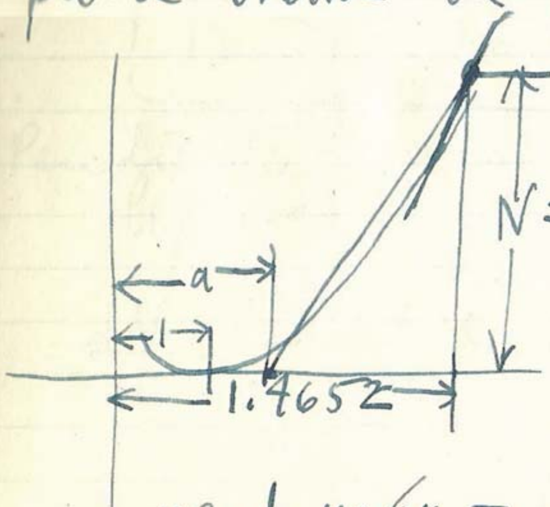
$\log N = 8.99815$

$N = 0.099575$

195
6.6
6.6
202.3

$\boxed{\begin{matrix} N = 0.099575 \\ v_2 = 1.4652 \end{matrix}}$ Punto de inflexión ($M=1$) correspondiente valor de $r = 5.23607$

~~Ma~~ Pendiente máxima en recta para órbita de tipo elíptico.



$$\text{pend. máx} = \frac{0.099575}{1.4652 - a}$$

$$\text{pend. máx} = \frac{0.099575}{1.4652 - 1 + 2E}$$

$$\frac{4M^2}{H^2} < \frac{0.099575}{0.4652 + 2E}$$

$$\frac{H^2}{4M^2} > \frac{0.4652 + 2E}{0.099575}$$

$$H > 2M\sqrt{\frac{2}{0.099575}}\sqrt{E + \frac{0.4652}{2}}$$

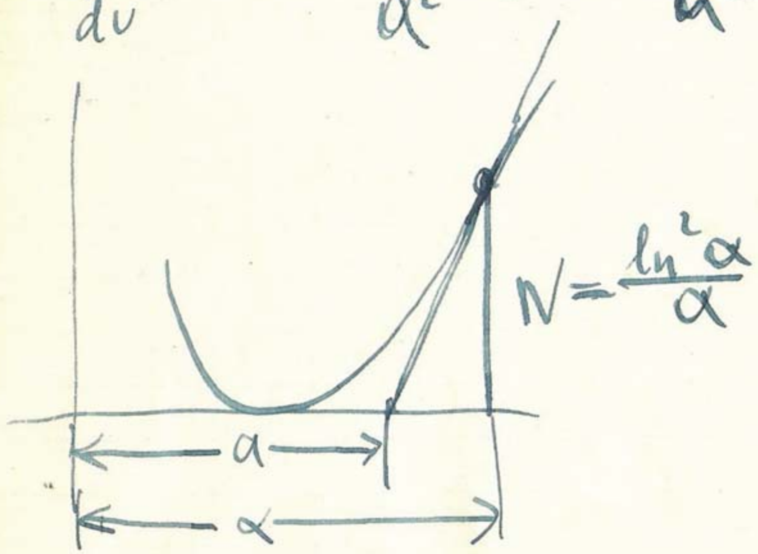
$$\log 2 = 0.30103$$
$$\log 0.099575 = 8.99815$$
$$\log(\text{subrad.}) = 1.30288$$
$$\log\sqrt{\ } = 0.65144$$
$$\log 2 = 0.30103$$
$$0.95247$$
$$8.9634$$

$$\boxed{H > 8.9634\, M\sqrt{E + 0.2326}}$$

Pendiente mínima de recta para tipo elíptico.

$$\frac{dN}{d\upsilon} = -\frac{\ln^2\alpha}{\alpha^2} + \frac{2\ln\alpha}{\alpha^2}$$

$$\frac{N}{\alpha - a} = -\frac{\ln^2\alpha}{\alpha^2} + \frac{2\ln\alpha}{\alpha^2}$$



$$\frac{\frac{\ln^2\alpha}{\alpha}}{\alpha - a} = -\frac{\ln^2\alpha}{\alpha^2} + \frac{2\ln\alpha}{\alpha^2}$$

$$\frac{\alpha\ln\alpha}{\alpha - a} = -\ln\alpha + 2$$

$$\frac{\alpha - a}{\alpha\ln\alpha} = \frac{1}{2-\ln\alpha}$$

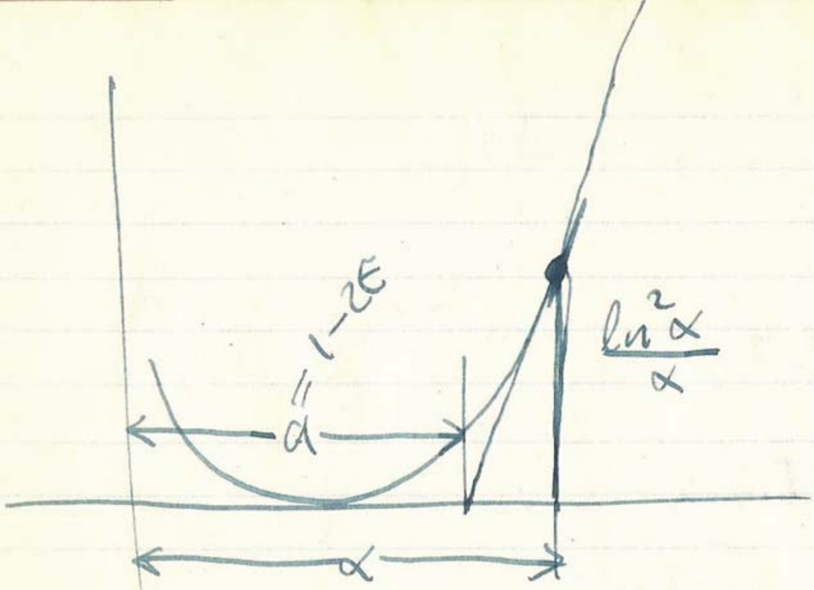
$$\alpha - a = \frac{\alpha\ln\alpha}{2-\ln\alpha}$$

$$\alpha - \frac{\alpha\ln\alpha}{2-\ln\alpha} = a = 1-2E$$

$$\frac{2\alpha - \alpha\ln\alpha}{2-\ln\alpha} = 1-2E$$

$$E = \frac{1}{2} - \frac{\alpha - \alpha\ln\alpha}{2-\ln\alpha}$$

$$\boxed{E = \frac{1}{2} - \alpha\,\frac{1-\ln\alpha}{2-\ln\alpha}}$$



$$\frac{dN}{dU} = -\frac{\ln^2\alpha}{\alpha^2} + \frac{2\ln\alpha}{\alpha^2} = \frac{\ln\alpha\,[2-\ln\alpha]}{\alpha^2}$$

$$\frac{\ln\alpha\,[2-\ln\alpha]}{\alpha^2} = \frac{\ln^2\alpha}{\alpha(\alpha-a)}$$

$$\frac{[2-\ln\alpha]}{\alpha} = \frac{\ln\alpha}{\alpha-a}$$

$$\alpha - a = \frac{\alpha\ln\alpha}{2-\ln\alpha}$$

$$a = \alpha - \frac{\alpha\ln\alpha}{2-\ln\alpha}$$

$$a = \frac{2\alpha - 2\alpha\ln\alpha}{2-\ln\alpha} = \frac{2\alpha[1-\ln\alpha]}{2-\ln\alpha}$$

$$1-2E = \frac{2\alpha[1-\ln\alpha]}{2-\ln\alpha} \qquad 2E = 1 - \frac{2\alpha[1-\ln\alpha]}{2-\ln\alpha}$$

$$\cancel{2E = \frac{2-\ln\alpha-2\alpha+2\alpha\ln\alpha}{2-\ln\alpha}}$$

$$E = \frac{1}{2} - \frac{\alpha\,[1 - \ln\alpha]}{2 - \ln\alpha}$$

$$E = \frac{1}{2} - \alpha\,\frac{2 - \ln\alpha - 1}{2 - \ln\alpha}$$

$$E = \frac{1}{2} - \alpha\left[1 - \frac{1}{2 - \ln\alpha}\right]$$

$$E = \frac{1}{2} - \alpha + \frac{\alpha}{2 - \ln\alpha}$$

$$E = \frac{2 - \ln\alpha - 2\alpha + 2\alpha\ln\alpha}{2(2 - \ln\alpha)}$$

$$E = \frac{1}{2} + \frac{-2\alpha + \alpha\ln\alpha + \alpha}{2 - \ln\alpha}$$

$$E = \frac{1}{2} + \frac{-\alpha + \alpha\ln\alpha}{2 - \ln\alpha}$$

$$\frac{4M^2}{H^2} > \frac{\ln\alpha\left[2-\ln\alpha\right]}{\alpha^2}$$

$$H < \frac{2M\alpha}{\sqrt{\ln\alpha\left[2-\ln\alpha\right]}}.$$

$$E+\alpha-\frac{1}{2} = \frac{\alpha}{2-\ln\alpha}$$

$$2-\ln\alpha = \frac{\alpha}{E+\alpha-\frac{1}{2}}$$

$$\ln\alpha = 2-\frac{\alpha}{E+\alpha-\frac{1}{2}} = \frac{2E+2\alpha-1-\alpha}{E+\alpha-\frac{1}{2}}$$

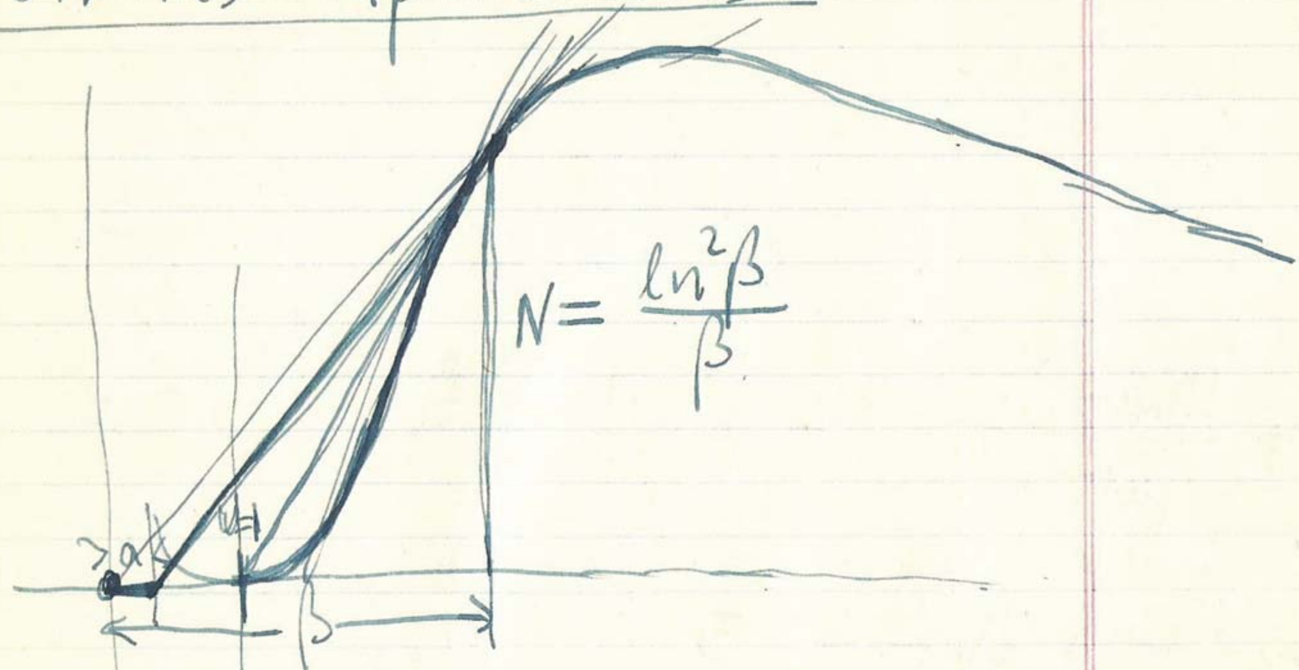
$$\ln\alpha = \frac{2E+\alpha-1}{E+\alpha-\frac{1}{2}}$$

$$H < \frac{2M\alpha}{\sqrt{\frac{2E+\alpha-1}{E+\alpha-\frac{1}{2}}\cdot\left[\frac{\alpha}{E+\alpha-\frac{1}{2}}\right]}}$$

$$H < \frac{2M\alpha\left(E+\alpha-\frac{1}{2}\right)}{\sqrt{\alpha(2E+\alpha-1)}}$$

$$H < 2M\left(E+\alpha-\frac{1}{2}\right)\sqrt{\frac{\alpha}{2E+\alpha-1}}$$

Orbitas hiperbólicas.



$$\cancel{\frac{4M^2}{V^2} < \frac{\ln^2\beta}{\beta(\beta-a)}}$$

~~H > M~~

$$-\frac{\ln^2\beta}{\beta^2} + \frac{2\ln\beta}{\beta^2} = \frac{\ln^2\beta}{\beta(\beta-a)}$$

$$-\frac{\ln\beta}{\beta} + \frac{2}{\beta} = \frac{\ln\beta}{\beta-a}$$

$$\frac{\beta-a}{\ln\beta} = \frac{\beta}{2-\ln\beta}$$

$$\boxed{\beta-a = \frac{\beta\ln\beta}{2-\ln\beta}}$$

$$\frac{\beta-a}{\ln\beta} = \frac{\beta}{2-\ln\beta}$$

$$\beta-a = \frac{\beta\ln\beta}{2-\ln\beta}$$

$$a = \frac{2\beta-2\beta\ln\beta}{2-\ln\beta} = \frac{2\beta-2\beta\ln\beta}{2-\ln\beta}$$

$$1-2E=\frac{2\beta-2\beta\ln\beta}{2-\ln\beta}$$

$$E=\frac{1}{2}-\frac{\beta-\cancel{2}\beta\ln\beta}{\cancel{2}(2-\ln\beta)}$$

$$E=\frac{1}{2}-\frac{\beta(1-\ln\beta)}{2-\ln\beta}=\frac{1}{2}-\frac{\beta(2-\ln\beta)}{2-\ln\beta}+\frac{\beta}{2-\ln\beta}$$

pendiente $=\frac{4M^2}{H^2}$

$$\boxed{E=\frac{1}{2}-\beta+\frac{\beta}{2-\ln\beta}}$$

$$\frac{4M^2}{H^2}<\frac{\frac{\ln^2\beta}{\beta}}{\beta-a}$$

$$\frac{H^2}{4M^2}>\frac{\beta(\beta-a)}{\ln^2\beta}$$

$$\frac{H^2}{4M^2}>\frac{\beta^2\ln\beta}{(2-\ln\beta)\ln^2\beta}$$

$$\frac{H^2}{4M^2}>\frac{\beta^2}{\ln\beta(2-\ln\beta)}$$

Máximo de $\cancel{N = \frac{\ln w}{w^2}}$

$$N = \frac{\ln^2 w}{w}$$

$$\frac{dN}{dw} = \frac{2\ln w}{w^2} - \frac{\ln^2 w}{w^2}$$

$$\frac{dN}{dw} = \frac{\ln w}{w^2}\left(2 - \ln w\right)$$

$$\frac{\ln w}{w^2}\left(2 - \ln w\right) = 0$$

$$\ln w = 2$$

$$w = e^2 = 7.389056$$

0.4342 94 48
0.86858.8 96
5.9

7389
738898

2.718 28 183   4
2.718 28 183   4
8 154 84 549   3
2174 62 54 64   5
271828 18 3   4
2174 6254 64   5
54365636 6.   8
2 17462 5464   5
2 71828183   4
190 279728 1   1
543 656366   8
738 9056 10730 81 489   7

$\underline{\underline{7.389056}}$

Máximo en $w = 7.389056 = e^2$

$$N = \frac{4}{7.389056} = \frac{4}{e^2}$$

$\log 4 = 0.60206$

$\log(e^2) = 0.86859$

$\log N = 9.73347$

0.54134

$N = 0.54134$

0.135

540

$$\boxed{\begin{array}{l}\text{Máximo en } w = 7.389056 \\ N = 0.54134\end{array}}$$

Tangente a la curva $N = \frac{\ln^2 w}{w}$

apoyada en el punto $(1,0)$

$$y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)\,w + \frac{2\ln w_0}{w_0}(\ln w_0 - 1)$$

$$\frac{\ln w_0}{w_0^2}(2 - \ln w_0) + \frac{2\ln w_0}{w_0}(\ln w_0 - 1) = 0$$

$$\frac{2 - \ln w_0}{w_0} = \cancel{2(\ln w_0 - 1)}\ 2(1 - \ln w_0)$$

$$\cancel{2.1}\quad \boxed{\frac{2 - \ln w_0}{1 - \ln w_0} = 2w_0}$$

$$\frac{\ln w_0}{2-\ln w_0} = \frac{w_0 - 1}{w_0}$$

$$\frac{2-\ln w_0}{2(1-\ln w_0)} = w_0$$

$$\frac{2-\ln w_0}{1-\ln w_0} = 2w_0$$

$$z = \frac{2-\ln w_0}{1-\ln w_0} - 2w_0$$

w_0	z
1	0
e	∞
e^2	$-2e^2$
e^3	

$$\frac{2-1}{1-1} - 2e$$

$$\frac{2-2}{1-2} - 2e^2$$

$$\frac{2-3}{1-3} - 2e^3 = +\frac{1}{2} - 2e^3$$

$$w = e(1+\varepsilon)$$

$$z = \frac{2-1-\ln(1+\varepsilon)}{1-1-\ln(1+\varepsilon)} - 2e(1+\varepsilon)$$

$$z = \frac{1-\varepsilon}{-\varepsilon} - 2e(1+\varepsilon)$$

$$z = -\frac{1}{\varepsilon} + 1 - 2e(1+\varepsilon)$$

$$\cancel{\ln e}\quad \frac{\ln\left(\frac{w_0}{e^2}\right)}{\ln\left(\frac{w_0}{e}\right)} = 2w_0$$

$$\frac{\log \frac{w_0}{e^2}}{\log \frac{w_0}{e}} = 2w_0$$

$$\frac{\log w_0 - 0.86859}{\log w_0 - 0.43429} = 2w_0$$

Tangente a la curva $y = \frac{\ln^2 w}{w}$ apoyada en el punto $(1,0)$

$$\frac{dy}{dw} = \frac{2\ln w}{w^2} - \frac{\ln^2 w}{w^2} = \frac{\ln w}{w^2}[2 - \ln w]$$

$$y - \frac{\ln^2 w_0}{w_0} = \frac{\ln w_0}{w_0^2}[2 - \ln w_0](w - w_0)$$

$$-\frac{\ln^{\cancel{2}} w_0}{\cancel{w_0}} = \frac{\cancel{\ln w_0}}{w_0^{\cancel{2}}}[2 - \ln w_0](1 - w_0)$$

$$-\frac{\ln w_0}{2 - \ln w_0} = \frac{1 - w_0}{w_0}$$

$$\cancel{\frac{\ln w_0}{2 - \ln w_0} = w_0 - 1}$$

$$\cancel{\frac{2 - \ln w_0 + \ln w_0}{2 - \ln w_0} = w_0 \qquad w_0 = \frac{2}{2 - \ln w_0}}$$

$$(2-\ln w_0) = 2w_0(1-\ln w_0)$$

$$z = (2-\ln w_0) - 2w_0(1-\ln w_0)$$

w_0	z
1	0
e	1
e^2	$2e^2$

$$1 - 2e(1-1)$$

$$(2-2) - 2e^2(1-2)$$

$$0 + \underline{2e^2}$$

Establecer la ecuación de la tangente a la curva

$$y=\frac{\ln^2 w}{w}$$

apoyada en el punto (1,0).

$$y=\frac{\ln^2 w}{w}$$

$$\frac{2\cancel{w}\ln w\cdot\frac{1}{\cancel{w}}-\ln^2 w}{w^2}$$

$$\frac{dy}{dw}=\frac{2\ln w}{w^2}-\frac{\ln^2 w}{w^2}=\frac{\ln w}{w^2}(2-\ln w)$$

$$y-y_0=\frac{\ln w_0}{w_0^2}(2-\ln w_0)(w-w_0)$$

$$\cancel{y_0=}$$

$$-\frac{\ln^2 w_0}{w^0}=\frac{\ln w_0}{w_0^2}(2-\ln w_0)(1-w_0)$$

$$-\ln w_0=\frac{1}{w_0}(2-\ln w_0)(1-w_0)$$

$$-\frac{\ln w_0}{2-\ln w_0}=\frac{1-w_0}{w_0}$$

$$\frac{\ln w_0}{2-\ln w_0}=\frac{w_0-1}{w_0}$$

$$w_0=\frac{2-\ln w_0}{2-2\ln w_0}=\frac{2-\ln w_0}{2(1-\ln w_0)}$$

$$\frac{\ln w_0}{2}=\frac{w_0-1}{2w_0-1}$$

$\log 1.9 = 0,27875$

$$\frac{1.9-1}{3.8-1} = \frac{0.9}{2.8}$$

$$\begin{array}{r|l} 2.8 & 0.78 \quad 731 \\ & 0.27919 \end{array}$$

22
257
53
251

$$\begin{array}{r} 0,27875 \\ 0,27919 \\ \hline 0.00044 \end{array}$$

para $w_0 = 1.9$ $z = -0.00044$

para $w_0 = 2.0$ $z = +0.01150$

0.01194

100 — 1194

$$\frac{100 \cdot 44}{1194}$$

$$1194 \overline{)4400} \quad \cancel{4}\ 38$$

~~824~~

-9180

$$\boxed{\ln w_0 = \frac{2(w_0-1)}{2w_0-1}}$$

2.30258 5
4.60517

$$\log w_0 = \cancel{4.60517}\ \frac{w_0-1}{2w_0-1}$$

0.434294448
0.868588896

w_0	$\log w_0$	s
~~2~~	~~0.30103~~	~~1.53506~~
~~4~~	~~0.60206~~	

$\frac{1}{3}$

$\frac{3}{7}$

$$\log w_0 = 0.86859\ \frac{w_0-1}{2w_0-1}$$

$\frac{2}{5} = 0.4$

$$\frac{\frac{3}{2}-1}{3-1} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

w_0	$\log w_0$	s
2	0.30103	0.28953
3	0.47712	0.347436
$\frac{3}{2}=1.5$	0.17609	0.21715

raíz entre 1.5 y 2

$$z = \log w_0 - 0.86859\ \frac{w_0-1}{2w_0-1}$$

w_0	z
1.5	−0.04
2	+0.01
1.9	

5
0.5

$w_0 = 1.9038$

$\log w_0 = 0.27\ 962$

$$\frac{w_0 - 1}{2w_0 - 1} = \frac{0.9038}{2.8076}$$

log 0.90 38	9.95 607
log 0.86 859	9.93 882
log numerador	9.89 489
log 2.80 76	0.44 834
log quebrado	9.44 655
quebrado	0.27961

$$\boxed{w_0 = 1.9038} \qquad N = 0.21775$$

$$N = \frac{\ln^2 w_0}{w_0}$$

$\log w_0 = 0.27962$

$\log(\log w_0) = 9.44\ 657$

$\log 0.43429448 = 9.63\ 778$

$= 9.80\ 879 = \log(\ln w_0)$

$\cancel{\log w_0 = 0.27\ 962}$

$\cancel{9.52\ 917}$

$\log(\ln^2 w_0) = 9.61\ 758$

$\log w_0 = 0.27\ 962$

$9.33\ 796$

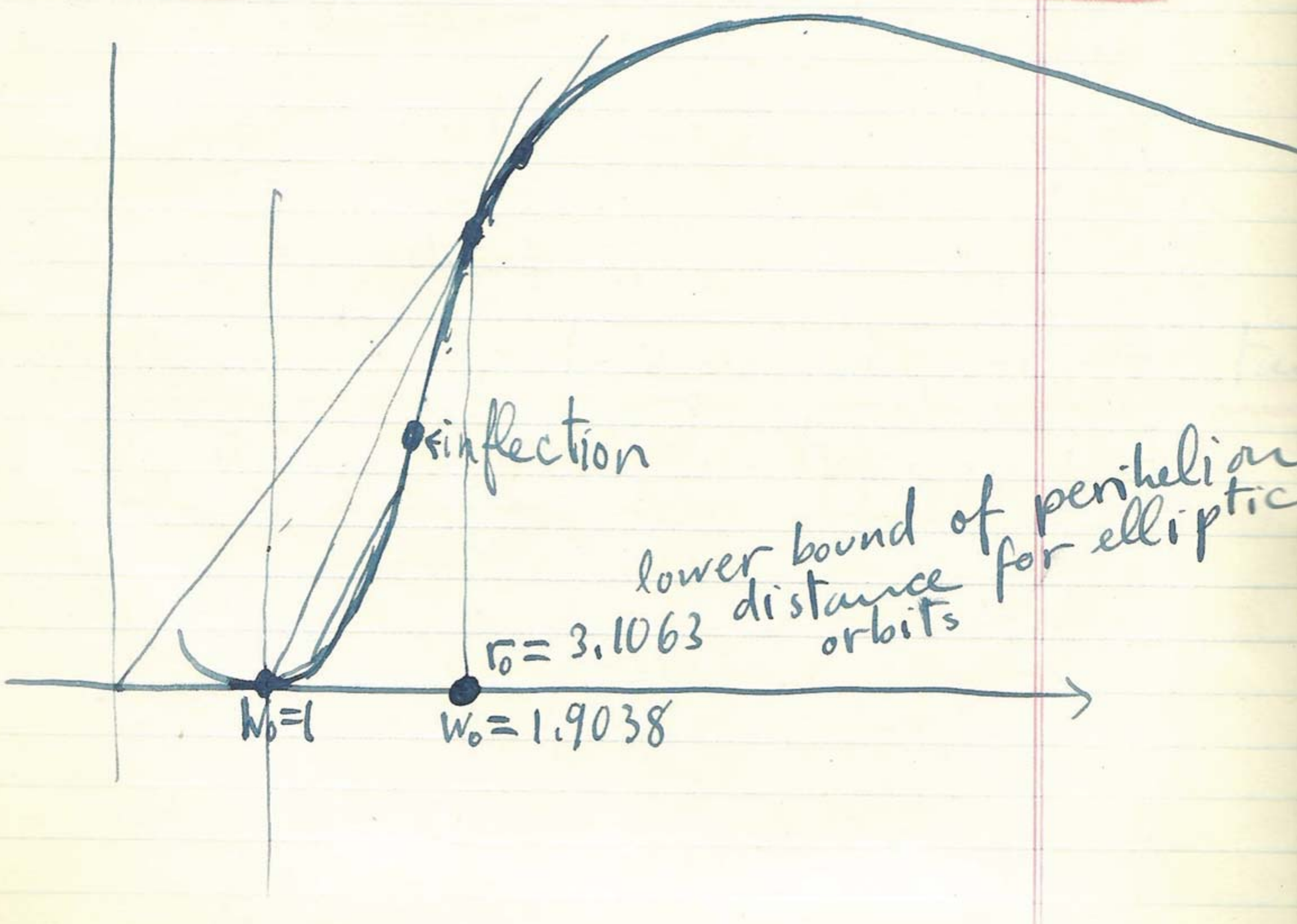
0.21775

lower bound for the perihelion distance of ~~elliptic orb.~~ orbits of elliptic type.

$$v_0 = \frac{1}{2} \ln w_0$$

$$r_0 = \frac{2}{\ln w_0}$$

$\log 2$	0.30103
$\log(\ln w_0)$	9.80879
$\log r_0$	0.49224
r_0	3.1063



Tangente a la curva $\frac{\ln^2 w}{w}$ apoyada en el punto (0,0).

$$y - \frac{\ln^2 w_0}{w_0} = \frac{\ln w_0}{w_0^2}(2-\ln w_0)(w-w_0)$$

$$+\frac{\ln^2 w_0}{\cancel{w_0}} = +\frac{\cancel{\ln w_0}}{\cancel{w_0^2}}(2-\ln w_0)\cancel{w_0}$$

$$\ln w_0 = 2 - \ln w_0$$

$$2\ln w_0 = 2$$

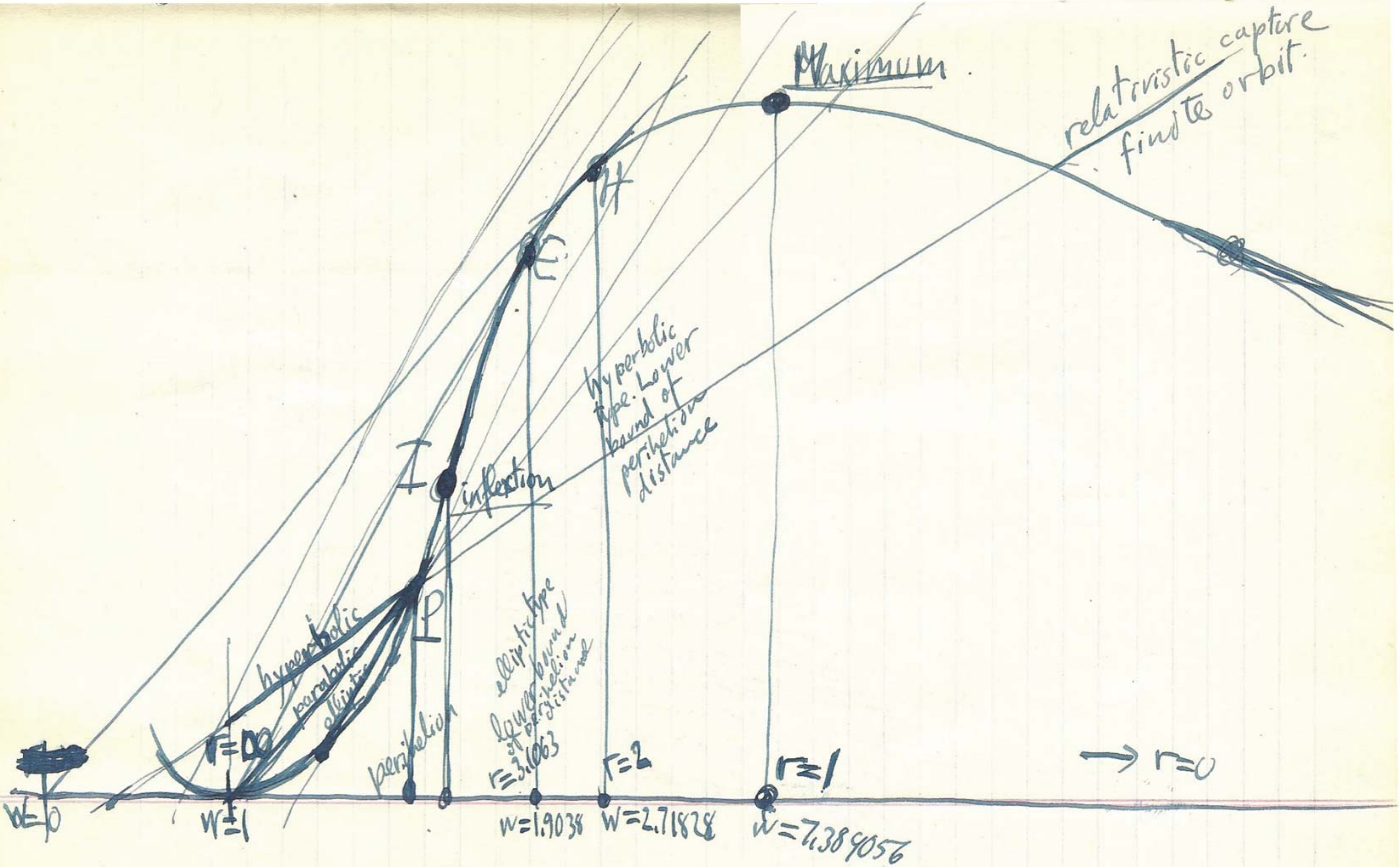
$$\ln w_0 = 1$$

$$\underline{\underline{w_0 = e}} = 2.71828182 8$$

$$2v = 1 \quad v = \frac{1}{2}$$

$$\underline{\underline{r_0 = 2}}$$

lower bound for the perihelion distance of orbits of hyperbolic type. $r_0=2$


Maximum
relativistic capture
find to orbit.
hyperbolic
parabolic
elliptic
perihelion
inflection
elliptic type
lower bound
of perihelion
distance
hyperbolic
type. Lower
bound of
perihelion
distance
r=∞
r=3.1063
r=2
r=1
→ r=0
W=0
W=1
W=1.9038
W=2.71828
W=7.389056

Maximum of the curve $N = \frac{\ln^2 w}{w}$.

$$\frac{2 \ln w}{w^2} - \frac{\ln^2 w}{w^2} = 0$$

$$2 - \ln w = 0 \qquad \ln w = 2$$

$$\boxed{w = e^2}$$

2.71 828 182 8

2

```
                      2.71  828  182 8
                      2.71  828  182 8
                      2174  625  462 4   7
                     5436   563  656     4
                  217462    546  24      7
                  271828    182  8       2
            21746254        624          7
            54365636        56           4
        2174625462          4            7
        2718281828                       2
      19027972796                        5
      5436563656                         4
      7.38905609  6435021   584          4
```

$$\boxed{7.389056} \qquad \boxed{e^2}$$

$$N = \frac{4}{e^2}$$

$$\begin{array}{r|l} & 0.541341 \\ \hline 7\,38\,90\,56 & 4\,00\;00\;00 \\ & 400\;00\;000 \\ & 30\;54\;7200 \\ & 099\;09760 \\ & 25\;207040 \\ & 30398720 \\ & -8424960 \end{array}$$

Máximo de la curva $\frac{\ln^2 w}{w}$ está en

$w = 7.38\,90\,56 = e^2$

$N = 0.54\,13\,41$

$w = e^{2v}$

$2v = 2$

$v = 1 \quad r = 1$

Inflections of the curve $\frac{\ln^2 w}{w}$

$$\frac{dN}{dw} = \frac{2\ln w}{w^2} - \frac{\ln^2 w}{w^2}$$

$$\frac{d^2N}{dw^2} = \frac{2}{w^3} - \frac{4\ln w}{w^3} - \frac{2\ln w}{w^3} + 2\frac{\ln^2 w}{w^3}$$

$$\frac{d^2N}{dw^2} = \frac{2 - 6\ln w + 2\ln^2 w}{w^3}$$

$$\ln^2 w - 3\ln w + 1 = 0$$

0.38 19 66
0.38 19 66

0.38 19 66 6
0.38 19 66 6
2 29 17 96 0
22 91 79 6 0
3 43 76 94 0
3 81 96 6 6
3 05 57 28 3
1 45 89 8 0
14 58 98 02 51 56 0

0.0995782

8

1.4651 | 0.1458 98 8
1 40 390 8
8 5310 8
~~1 1 2550~~
1 20550 4
33420

$$\ln w = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\ln w = \frac{3 \pm \sqrt{5}}{2}$$

$$\sqrt{5} = 2.2360680$$

$$\ln w_2 = \frac{5.2360680}{2}$$

$$\ln w_1 = \frac{0.7639320}{2}$$

$$\ln w_1 = 0.38\ 19\ 66$$

$$\ln w_2 = 2.61\ 80\ 34$$

```
        0.38 19 66        6
        0.43 42 94        8
     ---------------------------
        1 52 78 64        6
        7 37 69 4         0
      7 63 9 32           3
    1 52 78 64            6
   1 14 58 98             0
  1 52 78 64              6
  ------------------------------
 0.16 58 58 54 20 04      3
```

$$\log w_1 = 0.16\ 58\ 59$$

$\boxed{w_1 = 1.4651}$ $\boxed{N = 0.09\ 95\ 78}$

Inflection

$\ln w_2 = 2.6180340$

5.2360680

2. 61 80 34

0. 43 42 94

2. 61 80 34 (6)

0. 43 42 94 (8)

4 — 10 47 21 36 (6)

9 — 2 35 62 30 6 (0)

2 — 5 23 60 68 (3)

4 — 1 04 72 13 6 (6)

3 — 7 85 41 02 (0)

4 — 1 04 72 13 6 (6)

1. 13 69 96 45 79 96 (3)

$\log w_2 = 1.13700$ 27.6

1. 136996

$\boxed{\cancel{w_2 = 14.028}}$ $\boxed{w_2 = 13.709}$

13 672

2. 61 80 34 (6)

2. 61 80 34 (6)

10 47 21 36 (6)

78 54 10 2 (0)

2 09 44 27 2 (3)

2 61 80 34 (6)

1 57 08 20 4 (0)

5 23 60 68 (3)

6 85 41 02 02 51 56 (0)

$$\begin{array}{r|l} & 0.4999 \\ 13.709 & 6.85\ 41\ 02 \\ & 137\ 05\ 0 \\ & 13\ 66\ 92 \\ & 1\ 33\ 110 \end{array}$$

Curva $y = \frac{\ln^2 w}{w}$

Punto	w	y
mínimo	1	0
Máximo	7.389	0.5413
Primera inflexión	1.465	0.0996
Segunda inflexión	13.709	0.5000
Frontera inferior de perihelios órbitas elípt.	1.9038	0.21775
Frontera inferior de los perihelios de las órbitas hiperbólicas	2.71828	0.36788

0.43 429448
0.56 570552

Curva $y = \frac{\ln^2 w}{w}$.

Punto	Letra	w	y
mínimo	m	0.000	0.0000
máximo	M	7.389	0.5413

$$w = e^{\frac{2}{r}}$$

$$\cancel{\text{At}} \quad \frac{2}{r} = \ln w$$

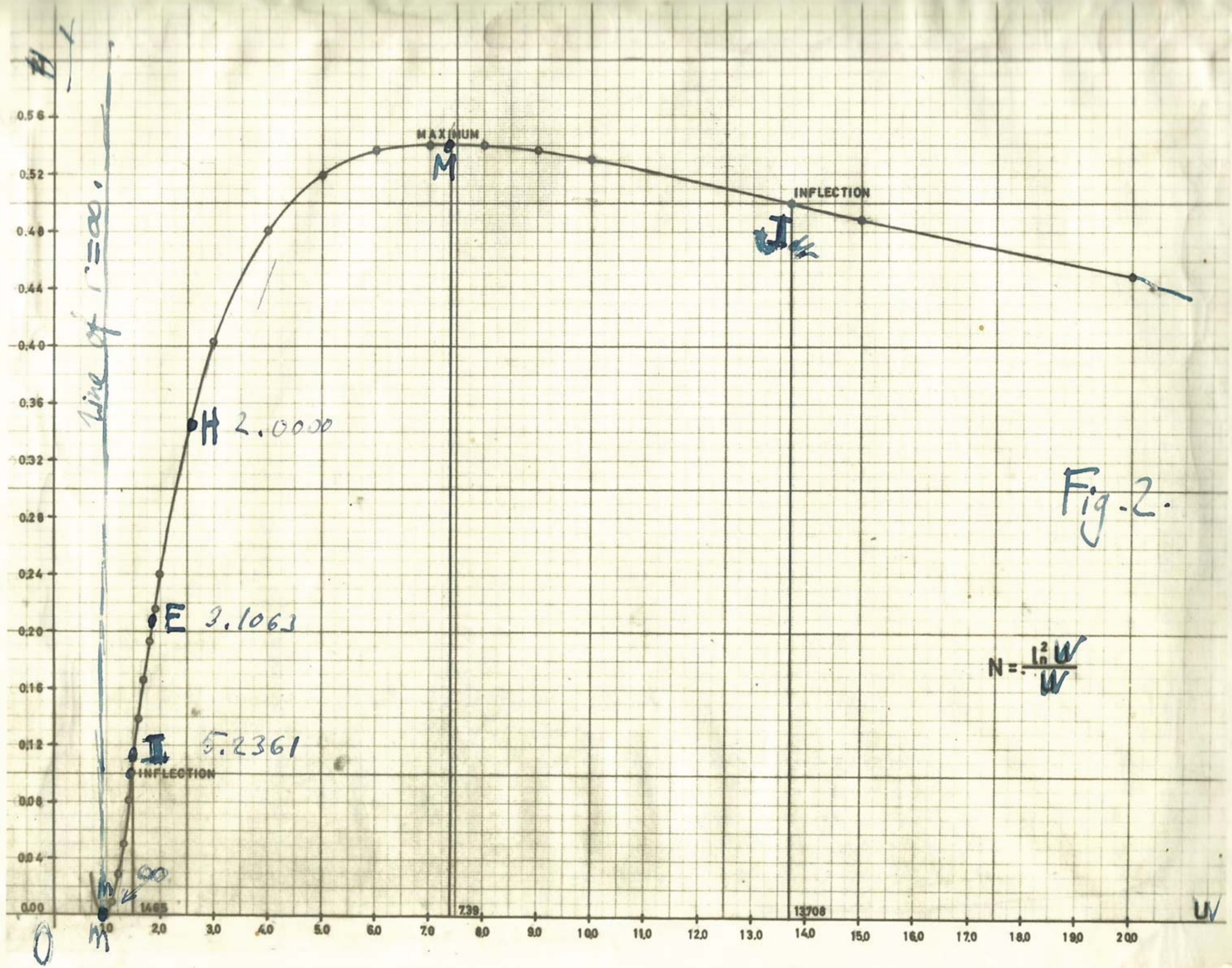
$$r = \frac{2}{\ln w}$$

```
                 5.236066
     6
.381966 ) 200 00 00      | 2
           9 01 700 0    | 8
           1 37 76 80    | 5
             23 17 820   | 5
               0 26 02400        2   5
                  ~~64706~~

                    0.763932
     6
.261 80 34 ) 2 00 00 00|0
              16 73 76 2 0        5
              1 02 94 1 6 0       5
                 24 400 580       5
                 - 8382740        5
                   - 5286380      5
```

Curva $y = \frac{\ln^2 w}{w}$

Punto	Símbolo	w	y	Símbolo	r
mínimo	m	0.0000	0.0000		∞
Primera inflexión	I	1.4650	0.0996		5.2361
Frontera inferior de los perihelios órbitas elípticas	E	1.9038	0.2178		3.1063
Frontera inferior de los perihelios de las órbitas hiperbólicas	H	2.7182	0.3679		2.0000
Máximo	M	7.3891	0.5413		1.0000
Segunda inflexión	J	13.709	0.5000		0.763932



Line of r = ∞.
MAXIMUM
INFLECTION
H 2.0000
E 3.1063
I 5.2361
INFLECTION
1465
7.39
13708
UV
N = l²n W / W

Fig. 2.

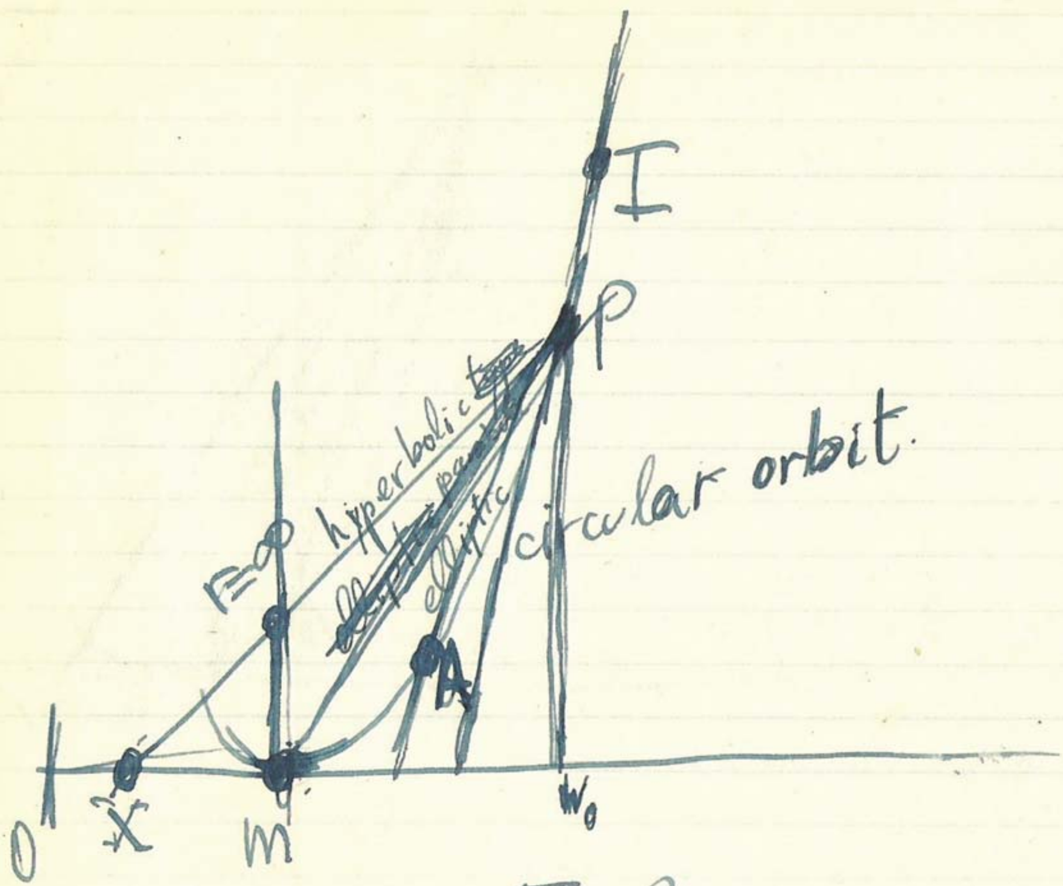


Fig. 3.

$$\frac{\ln^2 w_0}{w_0(w_0-1)} \text{ -- parabolic slope.}$$

$$0 < \frac{\ln^2 w_0}{w_0(w_0-1)} < \text{slope} \quad \text{elliptic type}$$

$$\frac{\ln^2 w_0}{w_0(w_0-1)} \doteq \text{slope} \quad \text{parabolic type}$$

$$\frac{\ln^2 w_0}{w_0(w_0-1)} > \text{slope} > \frac{\ln^2 w_0}{w_0} \quad \text{hyperbolic type}$$

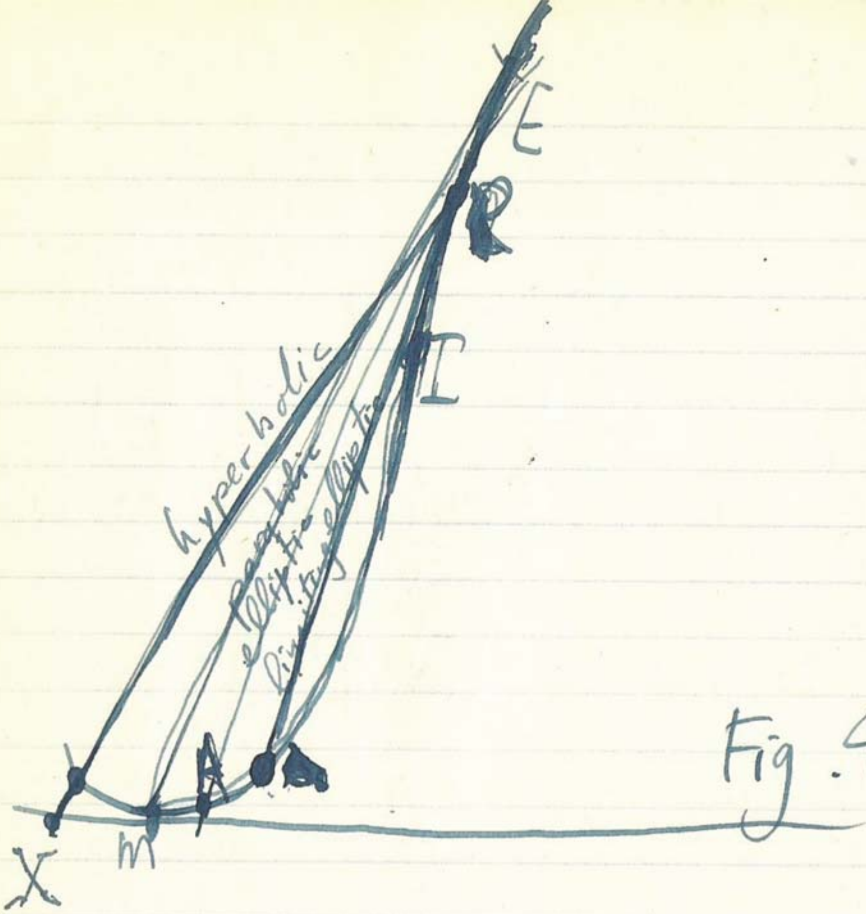


Fig. 4.

$a = 1 - 2E$

$s = \frac{4}{H^2}$

$H = e^{\frac{2}{r}} r^2 \dot{\theta}$

v_0 —— velocity at perihelion

$H = e^{\frac{2}{p}} p v_0 \qquad v_0 = \frac{e^{-\frac{2}{p}} H}{p} = \frac{H}{e^{\frac{2}{p}} p}$

Angular Momentum H.

1) $|H| = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}$; $\quad v_0 = \frac{1}{\sqrt{p-1}}$;

2) $|H| = p e^{\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$; $\quad v_0 = e^{-\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$;

3) $|H| = p e^{\frac{2}{p}}$; $\quad v_0 = 1$;

4) $|H| = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}$; $\quad v_0 = \frac{1}{\sqrt{p-1}}$;

5) $|H| = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}$; $\quad v_0 = \frac{1}{\sqrt{p-1}}$

6) $\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$;

7) $\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$;

8) $p e^{\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1} < |H| < p e^{\frac{2}{p}}$;

9) $\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{2}{p}}$.

---

(6) $\frac{1}{\sqrt{p-1}} < v_0 < e^{-\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$

(7) $\frac{1}{\sqrt{p-1}} < v_0 < e^{-\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1}$

(8) $e^{-\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1} < v_0 < 1$

$\cancel{e^{-\frac{1}{p}} \sqrt{e^{\frac{2}{p}} - 1} < v_0 < 1}$

$$\frac{\cancel{4}}{H^2} = \frac{\cancel{4}(p-1)}{p^2} e^{-\frac{4}{p}}$$

$$\boxed{H = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}} \qquad (1)$$

$$\frac{\cancel{4}}{H^2} = \frac{\cancel{4}}{p^2 e^{\frac{2}{p}}(e^{\frac{2}{p}}-1)}$$

$$\boxed{H = p e^{\frac{1}{p}} \sqrt{e^{\frac{2}{p}}-1}} \qquad (2)$$

$$\frac{\cancel{4}}{H^2} = \frac{\cancel{4}}{p^2 e^{\frac{4}{p}}}$$

$$\boxed{H = p e^{\frac{2}{p}}} \qquad (3)$$

$$\frac{\cancel{4}}{H^2} = \frac{\cancel{4}(p-1)}{p^2} e^{-\frac{4}{p}}$$

$$\boxed{H = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}} \qquad (4)$$

$$\boxed{H = \frac{p}{\sqrt{p-1}} e^{\frac{2}{p}}} \qquad (5)$$

---

(9) $$\frac{1}{\sqrt{p-1}} < V_0 < 1$$

$$\frac{4}{p^2 e^{\frac{2}{p}}(e^{\frac{2}{p}}-1)} < \frac{4}{H^2} < \frac{4(p-1)}{p^2} e^{-\frac{4}{p}}$$

$$\frac{p^2}{p-1} e^{\frac{4}{p}} < H^2 < p^2 e^{\frac{2}{p}}(e^{\frac{2}{p}}-1)$$

$$\boxed{\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{1}{p}}\sqrt{e^{\frac{2}{p}}-1}} \quad (6)$$

$$\boxed{\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{1}{p}}\sqrt{e^{\frac{2}{p}}-1}} \quad (7)$$

$$\frac{4}{p^2 e^{\frac{4}{p}}} < \frac{4}{H^2} < \frac{4}{p^2 e^{\frac{2}{p}}(e^{\frac{2}{p}}-1)}$$

$$p^2 e^{\frac{2}{p}}(e^{\frac{2}{p}}-1) < H^2 < p^2 e^{\frac{4}{p}}$$

$$\cancel{< |H| < p e^{\frac{1}{p}}\sqrt{e^{\frac{2}{p}}-1}}$$

$$\boxed{p e^{\frac{1}{p}}\sqrt{e^{\frac{2}{p}}-1} < |H| < p e^{\frac{2}{p}}} \quad (8)$$

$$\frac{\cancel{4}}{p^2 e^{\frac{4}{p}}} < \frac{\cancel{4}}{H^2} < \frac{4(p-1)}{p^2} e^{-\frac{4}{p}}$$

$$\cancel{\frac{1}{p^2 e^{\frac{4}{p}}}} \frac{p^2}{p-1} e^{\frac{4}{p}} < H^2 < p^2 e^{\frac{4}{p}}$$

$$\boxed{\frac{p}{\sqrt{p-1}} e^{\frac{2}{p}} < |H| < p e^{\frac{2}{p}}} \quad (9)$$

$$a = 1 - 2E$$

$$E = \frac{1-a}{2} = \frac{1}{2} - \frac{1}{2} a$$

$$\boxed{E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} \frac{p-2}{p-1}} \quad (1)$$

$$\boxed{E = 0} \quad (2)$$

$$\boxed{E = \frac{1}{2}} \quad (3)$$

$$\boxed{E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} \frac{p-2}{p-1}} \quad (4)$$

$$\boxed{E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} \frac{p-2}{p-1}} \quad (5)$$

$$E = \frac{1}{2} - \frac{1}{2}a$$

$$\boxed{\frac{1}{2} - \frac{1}{2}e^{\frac{2}{p}}\frac{p-2}{p-1} < E < 0} \quad (6)$$

$$\boxed{\frac{1}{2} - \frac{1}{2}e^{\frac{2}{p}}\left(\frac{p-2}{p-1}\right) < E < 0} \quad (7)$$

$$\boxed{0 < E < \frac{1}{2}} \quad (8)$$

$$\boxed{\frac{1}{2} - \frac{1}{2}e^{\frac{2}{p}}\left(\frac{p-2}{p-1}\right) < E < \frac{1}{2}} \quad (9)$$

Perihelion Apsidal distance	Velocity
$p < 2$	
$2 < p < 3.1063$	$\frac{1}{\sqrt{p-1}} < v_0 < 1$ $0 < v_0 < \frac{1}{\sqrt{p-1}}$
$3.1063 < p < 5.2361$	$\sqrt{1-e^{-\frac{2}{p}}} < v_0 < 1$ Parabolic $\frac{1}{\sqrt{p-1}} < v_0 < \sqrt{1-e^{-\frac{2}{p}}}$ $0 < v_0 < \frac{1}{\sqrt{p-1}}$
$5.2361 < p < \infty$	$\sqrt{1-e^{-\frac{2}{p}}} < v_0 < 1$ $v_0 = \sqrt{1-e^{-\frac{2}{p}}}$ $\frac{1}{\sqrt{p-1}} < v < \sqrt{1-e^{-\frac{2}{p}}}$

Orbits

Orbits of relativistic capture.

Hyperbolic type | Apse is a perihelion
Orbits of relativistic capture | Apse is an aphelion

Hyperbolic type. | Apse is a perihelion.
Elliptic type. | Apse is a perihelion.
Relativistic capture | Apse is an aphelion.

Hyperbolic type. | Apse is ~~ab~~ perihelion
Parabolic orbit. | Apse is a perihelion
circular
Elliptic type | Apse is a perihelion

$$\frac{w}{w^2\ln^4 w} + \frac{H}{8}\cdot\frac{1}{w}\left[\frac{4\ln w}{4w} - \frac{4}{H^2}w + \frac{4(1-2E)}{H^2}\right]=0$$

$$\frac{2\dot{w}}{w^2\ln^4 w} + \frac{H^2}{8w}\left[\frac{\ln^2 w}{w} - \frac{4}{H^2}w + \frac{4(1-2E)}{H^2}\right]=0$$

Analizar los casos en que $\dot{r}=0$ y

Ecuación (31)

$$\frac{1}{2}\dot{r}^2 + \frac{H^2}{2r^2}e^{-\frac{4}{r}} - \frac{1}{2} + \frac{(1-2E)}{2}e^{-\frac{2}{r}} = 0$$

$$\dot{r}\ddot{r} - \frac{H^2}{r^3}\dot{r}e^{-\frac{4}{r}} + \frac{2H^2}{r^4}\dot{r}e^{-\frac{4}{r}} + (1-2E)\frac{\dot{r}}{r^2}e^{-\frac{2}{r}} = 0$$

$$\cancel{\dot{r} = \frac{H}{2}\sqrt{e}}$$

$$\dot{r}^2 = 1 - \frac{H^2}{r^2} e^{-\frac{4}{r}} - (1-2E)\, e^{-\frac{2}{r}}$$

$$\boxed{\dot{r} = \sqrt{1 - \frac{H^2}{r^2} e^{-\frac{4}{r}} - (1-2E)\, e^{-\frac{2}{r}}}}$$

$$\ddot{r} = \frac{1}{2\dot{r}}\left[\frac{2H^2}{r^3}\dot{r}\, e^{-\frac{4}{r}} - \frac{4H^2\dot{r}}{r^4} e^{-\frac{4}{r}} - \frac{2\dot{r}}{r^2}(1-2E)\right]$$

$$\boxed{\ddot{r} = \frac{H^2}{r^3} e^{-\frac{4}{r}} - \frac{2H^2}{r^4} e^{-\frac{4}{r}} - \frac{1}{r^2}(1-2E)}$$

$$U'^2 + U^2 = \frac{1}{H^2}e^{4U} - \frac{1-2E}{H^2}e^{2U}$$

$$U'U'' + UU' = \frac{2}{H^2}e^{4U}U' - \frac{1-2E}{H^2}e^{2U}U'$$

$$\boxed{U'' = -U + \frac{2}{H^2}e^{4U} - \frac{1-2E}{H^2}e^{2U}}$$ Ecuación (20)

$$\boxed{U''' = -U' + \frac{8}{H^2}e^{4U}U' - \frac{2(1-2E)}{H^2}e^{2U}U'}$$

$$U^{IV} = \begin{cases} -U'' + \frac{8}{H^2}e^{4U}U'' - \frac{2(1-2E)}{H^2}e^{2U}U'' \\ + \frac{32}{H^2}e^{4U}U'^2 - \frac{4(1-2E)}{H^2}e^{2U}U'^2 \end{cases}$$

$$1-2E = -\frac{4e^{2U_0}U_0}{(1-U_0)} \qquad \Bigg\| \qquad \frac{1}{H^2} = \frac{U_0}{e^{4U_0}}(1-U_0)$$

$$U_0'' = -U_0 + \frac{2U_0}{\cancel{e^{4U_0}}}(1-U_0)\cancel{e^{4U_0}} + \frac{4e^{2U_0}U_0}{\cancel{1-U_0}} \cdot \frac{U_0}{\cancel{e^{4U_0}}}(\cancel{1-U_0})e^{2U_0}$$

$$U_0'' = -U_0 + 2U_0 - 2U_0^2 + 4U_0^2 = U_0 + 2U_0^2$$

$$y = \frac{\ln^2 w}{w}$$

$$\frac{dy}{dw} = \frac{2\ln w}{w^2} - \frac{\ln^2 w}{w^2} = \frac{\ln w}{w^2}\left[2 - \ln w\right]$$

en el punto w_0 $\qquad y_0 = \dfrac{\ln^2 w_0}{w_0}$

$$\left(\frac{dy}{dw}\right)_0 = \frac{\ln w_0}{w_0^2}\left[2 - \ln w_0\right]$$

Tangente:

$$y - \frac{\ln^2 w_0}{w_0} = \frac{\ln w_0}{w_0^2}\left[2 - \ln w_0\right](w - w_0)$$

$$\cancel{y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)w + \frac{\ln^2 w_0}{w_0} + \frac{\ln^2 w_0}{w_0^2}w_0}$$

$$y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)w \cancel{+ 2\frac{\ln^2 w_0}{w_0}}$$

$$\frac{4}{H^2} = \frac{\ln w_0}{w_0^2}(2 - \ln w_0) \qquad \Bigg\| \qquad \frac{1}{H^2} = \frac{\ln w_0}{4w_0^2}(2 - \ln w_0)$$

Como deben ser las constantes E y H para que el ápside sea transicional.

$$y = \frac{\ln^2 w}{w}$$

$$\frac{dy}{dw} = \frac{2\ln w}{w^2} - \frac{\ln^2 w}{w^2} = \frac{\ln w}{w^2}(2 - \ln w)$$

Ecuación de la tangente en $\left(w_0, \frac{\ln^2 w_0}{w_0}\right)$

$$\cancel{y = w_0} \quad y - \frac{\ln^2 w_0}{w_0} = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)(w - w_0)$$

$$y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)w + \frac{\ln^2 w_0}{w_0} - \frac{\ln w_0}{w_0}(2 - \ln w_0)$$

$$y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)w + \frac{\ln^2 w_0}{w_0} - 2\frac{\ln w_0}{w_0} + \frac{\ln^2 w_0}{w_0}$$

$$y = \frac{\ln w_0}{w_0^2}(2 - \ln w_0)w + 2\frac{\ln w_0}{w_0}(\ln w_0 - 1)$$

$$\left.\begin{aligned} \frac{4}{H^2} &= \frac{\ln w_0}{w_0^2}(2 - \ln w_0) \\ \frac{4(1-2E)}{H^2} &= 2\frac{\ln w_0}{w_0}(1 - \ln w_0) \end{aligned}\right\} \quad \text{Straight line (29)}$$

$$\boxed{H^2 = \frac{4w_0^2}{\ln w_0(2-\ln w_0)}}$$

¡Constantes críticas!

$$\boxed{\cancel{1-2E = 2\frac{w_0}{(2-\ln w_0)(1-\ln w_0)}}}$$

$$\boxed{1-2E = \frac{2w_0(1-\ln w_0)}{2-\ln w_0}}$$

$$v'' = -v + \frac{2}{H^2}e^{4v} - \frac{1-2E}{H^2}e^{2v}$$

$$H^2 = \frac{4e^{4v_0}}{2v_0(2-2v_0)} \qquad w_0 = e^{2v_0}$$

$$\boxed{H^2 = \frac{e^{4v_0}}{v_0(1-v_0)}}$$

$$1-2E = \frac{2e^{2v_0}(1-2v_0)}{2-2v_0} = \frac{e^{2v_0}(1-2v_0)}{1-v_0}$$

$$\boxed{(1-2E) = \frac{e^{2v_0}(1-2v_0)}{1-v_0}}$$

$$v_0'' = \begin{cases} -v_0 + 2v_0(1-v_0)\cancel{e^{-4v_0}}\cancel{e^{4v_0}} \cancel{- \frac{e^{2v_0}(1-2v_0)}{1-v_0}e^{2v_0}} \\ - \frac{e^{2v_0}(1-2v_0)}{\cancel{1-v_0}}e^{2v_0}\frac{v_0\cancel{(1-v_0)}}{\cancel{e^{4v_0}}} \end{cases}$$

$$v_0'' = -v_0 + 2v_0 - 2v_0^2 - v_0 + 2v_0^2 = 0$$

$$v_0'' = 0$$

$$\boxed{H = \frac{e^{2v_0}}{\sqrt{v_0(1-v_0)}}} \qquad E = \frac{1}{2} - \frac{(1-2v_0)}{2(1-v_0)}\, e^{2v_0}$$

Para estos valores de las constantes E y H y para las condiciones iniciales $v = v_0$ y $v' = 0$ se obtiene $v'' = 0$

$$v = v_0 \quad v' = 0 \quad v_0'' = 0, \quad v_0''' = 0 \; - - -$$

órbita circular.

La energía E y ~~el~~ momento angular H en función de la distancia apsidal y de la velocidad apsidal.

$$E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} (1 - v_0^2)$$

$$H = e^{\frac{2}{p}} p \, v_0$$

Órbitas circulares: $v_0 = \dfrac{1}{\sqrt{p-1}}$

Órbitas parabólicas: $v_0 = \sqrt{1 - e^{-2/p}}$

Curva de las órbitas circulares:

$$E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} \left(1 - \frac{1}{p-1}\right)$$

$$E = \frac{1}{2} - \frac{1}{2} e^{\frac{2}{p}} \left(\frac{p-1-1}{p-1}\right)$$

$$\frac{1}{2} - E = \frac{1}{2} e^{\frac{2}{p}} \frac{p-2}{p-1}$$

$$\frac{1-2E}{2} = \frac{1}{2} e^{\frac{2}{p}} \frac{p-2}{p-1}$$

$$\boxed{1 - 2E = e^{2/p} \frac{p-2}{p-1}} \quad \odot$$

$$\boxed{H = \frac{e^{\frac{2}{p}} p}{\sqrt{p-1}}} \quad \odot$$

$$\frac{1-2E}{H} = \frac{p-2}{p\sqrt{p-1}} \qquad p = 1+q^2$$

$$\frac{1-2E}{H} = \frac{1}{\alpha} \qquad \frac{q^2-1}{(1+q^2)\,q} = \frac{1}{\alpha}$$

$$q^3+q = \alpha q^2 - \alpha$$

$$\cancel{\alpha}\, p^2(p-1) = (p-2)^2\alpha^2 \qquad q^3-\alpha q^2+q+\alpha=0$$

$$q^2(q-\alpha)+(q+\alpha)=$$

$$p^3-p^2 = \alpha^2(p^2-2p+4)$$

$$p^3-(\alpha^2+1)p^2+2\alpha^2 p-4\alpha^2 = 0$$

$$p = 1+\sec^2\theta$$

$$p-1 = \sec^2\theta$$

$$p-2 = \sec^2\theta - 1 = \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta}$$

$$p-2 = \frac{\operatorname{sen}^2\theta}{\cos^2\theta} = \tan^2\theta$$

$$\frac{\tan^2\theta}{(1+\sec^2\theta)\sec\theta} = \frac{\operatorname{sen}^2\theta\cos\theta}{\cos^2\theta(1+\sec^2\theta)} = \frac{\operatorname{sen}\theta\tan\theta}{(1+\sec^2\theta)}$$

Espacio de cuatro dimensiones con g_{ij} constante.

$$ds^2 = g_{ij}\,dx^i\,dx^j \qquad i,j = 1,2,3,4.$$

$$g_{ij} = \text{constante}.$$

Curva definida por:

$$x^i = x^i(s).$$

$$\boxed{\bullet = \frac{d}{ds}}$$

Cuadrivector unitario tangente a la curva $t^i = \dot{x}^i$

$$\dot{x}^i = \frac{dx^i}{ds} = t^i$$

$$1)\quad \boxed{g_{ij}\,t^i t^j = 1}$$

Derívese ~~uso~~ (1) con respecto a s:

$$\underline{g_{ij}\,\dot{t}^i t^j = 0}$$

$$\dot{t}^i \text{ es } \perp \text{ a } t^i$$

porque $g_{ij}\,\dot{t}^i\,t^j = 0$.

La magnitud de $\dot{t}^i$ se designa con $\frac{1}{\rho}$.

$$g_{ij}\,\dot{t}^i\,\dot{t}^j = \cancel{\frac{1}{\rho^2}}\,\rho^2$$

$$\dot{t}_i\,\dot{t}^i = \cancel{\frac{1}{\rho^2}}\,\rho^2$$

Definición del cuadrivector normal principal n

$$\boxed{n^i = \frac{1}{\rho}\dot{t}^i}$$

$$g_{ij}\, n^i n^j = 1$$

$$g_{ij}\frac{\dot{t}^i}{\rho}\frac{\dot{t}^j}{\rho} = \frac{\rho^2}{\rho^2} = 1$$

El cuadrivector normal principal es unitario

El cuadrivector normal principal es $\perp$ al cuadrivector tangente.

$$g_{ij}\, t^i \underline{n^j = 0}\,.$$

$$n^i = \frac{1}{\rho}\dot{t}^i$$

Derívese con respecto a la longitud de arco s.

$$\cancel{\dot{n}^i = \ldots - \frac{\dot{\rho}}{\rho^2}\dot{t}^i}$$

$$\dot{n}^i = -\frac{\dot{\rho}}{\rho^2}\dot{t}^i + \frac{1}{\rho}\ddot{t}^i$$

$$n_i \dot{n}^i = -\frac{\dot{\rho}}{\rho^2}\rho n^i n_i + \frac{1}{\rho}\ddot{t}^i n_i$$

$$0 = -\frac{\dot{\rho}}{\rho} + \frac{1}{\rho} n_i \ddot{t}^i$$

$$\boxed{n_i \ddot{t}^i = \dot{\rho}}$$

$$n^i = \frac{1}{\rho}\dot{t}^i \therefore \dot{t}^i = \rho n^i$$

$$\dot{n}^i = -\frac{\dot{\rho}}{\rho^2}\dot{t}^i + \frac{1}{\rho}\ddot{t}^i$$

$$\dot{n}^i = -\frac{\dot{\rho}}{\rho} n^i + \frac{1}{\rho}\ddot{t}^i$$

$$\cancel{n_i \cdot \dot{n}^i} \quad n_i \dot{n}^i = -\frac{\dot{\rho}}{\rho} n_i n^i + \frac{1}{\rho}\ddot{t}^i n_i$$

$$\boxed{n_i \ddot{t}^i = \dot{\rho}}$$

$$\cancel{\dot{n}^i = \alpha \dot{t}^i + \beta b^i}$$

$$\dot{n}^i t_i = \cancel{\dot{\rho}} + \frac{1}{\rho}\ddot{t}^i t_i$$

$$\frac{1}{\rho}\ddot{t}^i = \frac{\dot{\rho}}{\rho} n^i + \alpha b^i$$

$$\frac{1}{\rho}\dot{t}_i \ddot{t}^i = \frac{\dot{\rho}}{\rho}\dot{t}_i n^i + \alpha \dot{t}_i b^i$$

tangente : t^1, t^2, t^3, t^4 ; t^i

normal principal : n^1, n^2, n^3, n^4 ; n^i

binormal : b^1, b^2, b^3, b^4 ; b^i

trinormal : h^1, h^2, h^3, h^4 ; h^i

$$h_i = \varepsilon_{ijkl}\, t^j n^k b^l$$

$$J_i = \varepsilon_{ijkl}\, x'^j x''^k x'''^l$$

$$\cancel{h_i = \frac{1}{\sigma \rho^4} J_i}$$

$$\boxed{h_i = \sigma \rho^2 J_i}$$

binormal (no unitaria): $x'^i + \rho\rho' x''^i + \rho^2 x^i$

normal unitaria: $n^i = \frac{1}{\rho} x''^i$

cuadrivector tangente unitario } $t^i = \overset{\bullet}{x}{}^i$; $\boxed{\bullet = \frac{\delta}{\delta s}}$.

$\boxed{\bullet = \frac{\delta}{\delta s}}$ Derivada intrínseca.

~~test~~ Cuadrivector normal unitario } $n^i = \frac{1}{\rho}\overset{\bullet\bullet}{x}{}^i = \frac{1}{\rho}\overset{\bullet}{t}{}^i$.

Derívese el cuadrivector normal unitario }

$$\overset{\bullet}{n}{}^i = -\frac{\overset{\bullet}{\rho}}{\rho^2}\overset{\bullet}{t}{}^i + \frac{1}{\rho}\overset{\bullet\bullet}{t}{}^i$$

$$\overset{\bullet}{t}{}^i = \rho\, n^i$$

$$\overset{\bullet}{n}{}^i = -\frac{\overset{\bullet}{\rho}}{\rho} n^i + \frac{1}{\rho}\overset{\bullet\bullet}{t}{}^i$$

Búsquese la componente de $\overset{\bullet}{n}{}^i$ $\perp$ a a t^i. ~~$\overset{\bullet}{n}{}^i$ es $\perp$ a $\overset{\bullet\bullet}{n}{}^i$~~

$$\overset{\bullet}{n}{}^i t_i = -\frac{\overset{\bullet}{\rho}}{\rho} n^i t_i + \frac{1}{\rho} t_i \overset{\bullet\bullet}{t}{}^i$$

$$\overset{\bullet}{n}{}^i t_i = \frac{1}{\rho} t_i \overset{\bullet\bullet}{t}{}^i$$

$$\overset{\bullet}{n}{}^i - (\overset{\bullet}{n}{}^j t_j)t^i = -\frac{\overset{\bullet}{\rho}}{\rho} n^i + \frac{1}{\rho}\overset{\bullet\bullet}{t}{}^i - \frac{1}{\rho}(t_j \overset{\bullet\bullet}{t}{}^j)t^i$$

Cálculo de $t_j \dddot{t}^j$.

$t_j \ddot{t}^j = 0$

$t_j \dddot{t}^j + \dot{t}_j \ddot{t}^j = 0$

$$\boxed{t_j \dddot{t}^j = -\frac{1}{\rho^2}}$$

Dirección de la binormal; binormal no unitaria $\left\{ \begin{array}{l} \cancel{-\dot{\rho} n^i + \rho \ddot{t}^i + \frac{1}{\rho^2} t^i} \\ -\rho^2 \dot{\rho} n^i + \rho^2 \ddot{t}^i + t^i \end{array} \right.$

Magnitud de ese cuadrivector $\left\{ \sqrt{\begin{array}{l} \rho^4 \dot{\rho}^2 + 1 + \rho^4 \|\ddot{t}^i\| \\ -2\rho^4 \dot{\rho} (n^i, \ddot{t}^i) + 2\rho^2 (\ddot{t}^i, t^i) \end{array}} \right.$

Magnitudes necesarias:

$\ddot{t}^i \ddot{t}_i$, $n^i \ddot{t}_i$, $t^i \ddot{t}_i$.

$t^i \ddot{t}_i$.

$t^i t_i = 1$

$\dot{t}^i t_i = \rho^2$

$$\boxed{\dot{t}^i \dot{t}_i = \rho^2}$$

$$2\dot{t}^i \ddot{t}_i = 2\rho\dot{\rho} \qquad \boxed{\dot{t}^i \ddot{t}_i = \rho\dot{\rho}}$$

~~Vec~~ Cuadrivector de posición x^i

Cuadrivector unitario tangente a la curva ~~unitar~~ $\dot{x}^i$

Cuadrivector normal principal $\ddot{x}^i$

$$\dddot{x}^i = \alpha \dot{x}^i + \beta \ddot{x}^i + \gamma b^i$$

$$\dot{x}_i \dddot{x}^i = \alpha$$

$$\ddot{x}_i \dddot{x}^i = \beta s_{22}$$

$$\dddot{x}^i \dddot{x}_i = \alpha^2 + \beta^2 s_{22} + \gamma^2$$

$$\dot{x}_i \ddot{x}^i = 0$$

$$\dot{x}_i \dddot{x}^i + \ddot{x}_i \ddot{x}^i = 0$$

$$\dot{x}_i \dddot{x}^i + \frac{1}{\rho^2} = 0$$

$$\boxed{\dot{x}_i \dddot{x}^i = -\frac{1}{\rho^2}}$$

$$\cancel{\ddot{x}_i \dddot{x}^i + \dot{x}_i \ddddot{x}^i}$$

$$\ddot{x}^i \ddot{x}_i = \frac{1}{\rho^2}$$

$$\dddot{x}^i \ddot{x}_i = \cancel{\ } - \frac{\dot{\rho}}{\rho^3}$$

~~Vec~~ Cuadrivector de posición $\}$ x^i

Cuadrivector unitario ~~normal~~ tangente a la curva $\}$: $\dot{x}^i = t^i$.

Notación: $s_{11} = \dot{x}_i \dot{x}^i = 1$. $\boxed{s_{11} = 1}$.

Cuadrivector normal principal (no unitario $\}$ $\ddot{x}^i = \dot{t}^i$.

$$\ddot{x}^i = \frac{1}{\rho} n^i$$

Cuadrivector unitario normal principal $\}$: n^i $\quad n^i = \rho \ddot{x}^i$

$$s_{22} = \ddot{x}_i \ddot{x}^i = \frac{1}{\rho^2} \qquad \boxed{s_{22} = \frac{1}{\rho^2}}$$

$$s_{12} = \dot{x}_i \ddot{x}^i = s_{21}$$

$$\dot{x}_i \dot{x}^i = 1$$

$$\dot{x}_i \ddot{x}^i = 0 \qquad \boxed{s_{12} = 0}$$

Definición del cuadrivector unitario binormal.

$b_i b^i = 1$.

$$\dot{n}^i = \alpha t^i + \beta b^i$$

Es obvio que $\dot{n}^i \perp n^i$.

El cuadrivector unitario binormal es un cuadrivector que cumple las siguientes condiciones:

1) es unitario $b^i b_i = 1$

2) es perpendicular al cuadrivector tangente y al cuadrivector normal principal

$$b^i n_i = 0$$
$$b_i t^i = 0$$

3) la derivada del cuadrivector normal principal está en el 3-espacio definido por el cuadrivector tangente, el cuadrivector normal y el cuadrivector normal principal.

$$\dot{n}^i = \alpha t^i + \beta b^i .$$

$$n^i = \rho \ddot{x}^i$$

$$\dot{n}^i = \dot{\rho} \ddot{x}^i + \rho \dddot{x}^i$$

$$\dot{n}^i = \frac{\dot{\rho}}{\rho} n^i + \rho \dddot{x}^i$$

$$\boxed{\dddot{x}^i = \frac{\alpha}{\rho} t^i + \frac{\beta}{\rho} b^i - \frac{\dot{\rho}}{\rho^2} n^i}$$

1) Cuadrivector de posición: x^i

2) Cuadrivector unitario tangente a la curva $\Big\}$ $t^i = \dot{x}^i = \dfrac{dx^i}{ds}.$

Notación $s_{11} = \dot{x}_i \dot{x}^i = t_i t^i = 1.$

3) Cuadrivector normal principal. (No unitario) $\Big\}$ $\ddot{x}^i = \dot{t}^i$

4) Cuadrivector unitario normal principal $\Big\}$ $n^i = \rho \ddot{x}^i$

$$\boxed{s_{22} = \ddot{x}_i \ddot{x}^i = \dot{t}_i \dot{t}^i = \frac{1}{\rho^2}}$$

$n^i n_i = 1.$

5) $s_{12} = \dot{x}_i \ddot{x}^i = ?$

$s_{11} = \dot{x}_i \dot{x}^i = 1$

$\dot{s}_{11} = \dot{x}_i \ddot{x}^i + \ddot{x}_i \dot{x}^i = 0$

$$\boxed{x_i \ddot{x}^i = 0}$$

$$\boxed{s_{12} = 0}$$

6) $\boxed{t_i n^i = 0}$ $\quad \boxed{t_i \ddot{x}^i = 0}$ $\quad \boxed{\dot{x}_i \ddot{x}^i = 0}$

7) $s_{12} = \dot{x}_i \ddot{x}^i = 0$

$\dot{s}_{12} = \dot{x}_i \dddot{x}^i + \ddot{x}_i \ddot{x}^i = 0$

$s_{13} + s_{22} = 0$

$s_{13} = - s_{22}$

$\cancel{s_{22}}$ $\boxed{s_{13} = - \frac{1}{\rho^2}}$

8) $s_{22} = \ddot{x}_i \ddot{x}^i = \frac{1}{\rho^2}$

$\dot{s}_{22} = 2\ddot{x}_i \dddot{x}^i = - \frac{2\dot{\rho}}{\rho^3}$

$\boxed{s_{23} = - \frac{\dot{\rho}}{\rho^3}}$

9) $s_{13} = \dot{x}_i \dddot{x}^i = - \frac{1}{\rho^2}$

$\dot{s}_{13} = \dot{x}_i \ddddot{x}^i + \ddot{x}_i \dddot{x}^i = + \frac{2\dot{\rho}}{\rho^3}$

$\boxed{s_{14} = + \frac{3\dot{\rho}}{\rho^3}}$

10) Definición del cuadrivector binormal unitario.

i) el cuadrivector ~~unit~~ binormal está en el 3 espacio definido por $\dot{x}^i$, $\ddot{x}^i$ y $\dddot{x}^i$

ii) el cuadrivector binormal es $\perp$ a t^i y a n^i

iii) el cuadrivector binormal es unitario.

De i) $$b^i = \alpha \dot{x}^i + \beta \ddot{x}^i + \gamma \dddot{x}^i$$

De ii) Multiplíquese b^i por $t^i = \dot{x}^i$ y por $\ddot{x}^i$

$$b^i \dot{x}_i = \alpha \dot{x}^i \dot{x}_i + \beta \ddot{x}^i \dot{x}_i + \gamma \dddot{x}^i \dot{x}_i$$

$$0 = \alpha s_{11} + \beta s_{12} + \gamma s_{13}$$

$$0 = \alpha - \frac{1}{\rho^2}$$

$$\alpha = \frac{1}{\rho^2}$$

$$b^i = \frac{1}{\rho^2} \dot{x}^i + \beta \ddot{x}^i + \gamma \dddot{x}^i$$

~~Multiplíquese b^i por b_i; se obtiene:~~

$$1 = \begin{cases} \frac{\gamma^2}{\rho^4} s_{11} + \frac{2\beta\gamma}{\rho^2} s_{12} + \frac{2\gamma^2}{\rho^2} s_{13} + \beta^2 s_{22} + 2\beta\gamma s_{23} \\ + \gamma^2 s_{33} \end{cases}$$

$$1 = \frac{\gamma^2}{\rho^4} - \frac{2\gamma^2}{\rho^4} + \frac{\beta^2}{\rho^2} - \frac{2\beta\gamma}{\rho^3}\dot{\rho} + \gamma^2 s_{33}$$

Multiplíquese por $\ddot{x}_i$.

$$0 = \frac{\gamma}{\rho^2} s_{12} + \beta s_{22} + \gamma s_{23}$$

$$0 = \frac{\beta}{\rho^2} - \gamma \frac{\dot{\rho}}{\rho^3}$$

$$\beta = \frac{\gamma\dot{\rho}}{\rho}$$

$$\cancel{b^i = \frac{\gamma}{\rho^2} \dot{x}^i + \frac{\gamma}{\rho} \ddot{x}^i + \gamma \dddot{x}^i}$$

Elévese al cuadrado; se obtiene:

$$1 = \left[\begin{array}{l} \frac{\gamma^2}{\rho^4} s_{11} + \frac{2\gamma^2}{\rho^3} s_{12} + \frac{2\gamma^2}{\rho^2} s_{13} \\ + \frac{\gamma^2}{\rho^2} s_{22} + 2\frac{\gamma^2}{\rho} s_{23} + \gamma^2 s_{33} \end{array} \right.$$

$$1 = \boxed{\frac{\gamma^2}{\rho^4} - \frac{2\gamma^2}{\rho^4} + \frac{\gamma^2}{\rho^4}} - \frac{2\gamma^2\dot{\rho}}{\rho^4} + \gamma^2 s_{33}$$

$0 = \alpha - \frac{\gamma}{\rho^2}$ $\quad \boxed{\alpha = \frac{\gamma}{\rho^2}}$

$$b^i = \frac{\gamma}{\rho^2}\dot{x}^i + \beta\ddot{x}^i + \gamma\dddot{x}^i$$ ¡Elévese al cuadrado!

$$b^i\ddot{x}_i = \frac{\gamma}{\rho^2}s_{12} + \beta s_{22} + \gamma s_{23}$$

$$0 = \frac{\beta}{\rho^2} - \frac{\gamma\dot{\rho}}{\rho^3}$$

$$\beta = \frac{\gamma\dot{\rho}}{\rho}$$

$$b^i = \frac{\gamma}{\rho^2}\dot{x}^i + \frac{\gamma\dot{\rho}}{\rho}\ddot{x}^i + \gamma\dddot{x}^i$$

Elévese al cuadrado:

$$1 = \frac{\gamma^2}{\rho^4}s_{11} + \cancel{\frac{2\gamma^2\dot{\rho}}{\rho^3}}s_{12} + \frac{2\gamma^2}{\rho^2}s_{13}$$
$$+ \frac{\gamma^2\dot{\rho}^2}{\rho^2}s_{22} + \frac{2\gamma^2\dot{\rho}}{\rho}s_{23} + \gamma^2 s_{33}$$

$$1 = \frac{\gamma^2}{\rho^4} - \frac{2\gamma^2}{\rho^4} + \frac{\gamma^2\dot{\rho}^2}{\rho^4} - \frac{2\gamma^2\dot{\rho}^2}{\rho^4} + \gamma^2 s_{33}$$

$$\boxed{s_{33} = \frac{1}{\gamma^2} + \frac{1}{\rho^4} + \frac{\dot{\rho}^2}{\rho^4}}$$

Defínase σ como sigue:

$$\gamma^2 = \sigma^2\rho^2 \qquad \gamma = \sigma\rho$$

$$s_{33} = \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\dot{\rho}^2}{\rho^4}$$

$$\boxed{b^i = \frac{\sigma}{\rho}\dot{x}^i + \sigma\dot{\rho}\,\ddot{x}^i + \sigma\rho\,\dddot{x}^i}$$

$$t^i = \dot{x}^i \qquad t^j = \dot{x}^j$$

$$n^j = \rho\,\ddot{x}^j \qquad n^k = \rho\,\ddot{x}^k$$

$$b^k \qquad b^l = \frac{\sigma}{\rho}\dot{x}^l + \sigma\dot{\rho}\,\ddot{x}^l + \sigma\rho\,\dddot{x}^l$$

$$h_i = \varepsilon_{ijkl}\, t^j n^k b^l$$

$$h_i = \sigma\rho^2 \varepsilon_{ijkl}\, \dot{x}^j \ddot{x}^k \dddot{x}^l$$

$$\boxed{h_i = \sigma\rho^2\, \varepsilon_{ijkl}\, \dot{x}^j \ddot{x}^k \dddot{x}^l}$$

Curvas en un espacio de cuatro dimensiones.

$ds^2 = dx^2 + dy^2 + dz^2 + dv^2$. Métrica.

$x = x(s)$; $y = y(s)$; $z = z(s)$; $v = v(s)$.

$$\boxed{{}' = \frac{d}{ds}}\ ,$$

$$s_{ij} = \frac{d^i x}{ds^i}\frac{d^j x}{ds^j} + \frac{d^i y}{ds^i}\frac{d^j y}{ds^j} + \frac{d^i z}{ds^i}\frac{d^j z}{ds^j} + \frac{d^i v}{ds^i}\frac{d^j v}{ds^j}.$$

$s_{11} = x'x' + y'y' + z'z' + v'v'$;

$s_{12} = x'x'' + y'y'' + z'z'' + v'v''$; etc.

$$\begin{Vmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{Vmatrix} = \begin{Vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & D \end{Vmatrix}$$

De $ds^2 = dx^2 + dy^2 + dz^2 + dv^2$;

se obtiene:

$1 = x'x' + y'y' + z'z' + v'v'$;

$$\boxed{s_{11} = 1}$$

$$s_{11} = x'x' + y'y' + z'z' + v'v'$$

Derívese con respecto a s:

$$x'x'' + y'y'' + z'z'' + v'v'' = 0$$

$$\boxed{s_{12} = 0} \qquad \boxed{s_{21} = 0}$$

$$A = 1; \quad H = 0$$

Definición: $\boxed{s_{22} = \frac{1}{\rho^2}}$

$$x''x'' + y''y'' + z''z'' + v''v'' = \frac{1}{\rho^2}$$

$$\boxed{B = \frac{1}{\rho^2}}$$

$$s_{11} = x'x' + y'y' + z'z' + v'v' = 1$$

$$s_{11}' = 2[x'x'' + y'y'' + z'z'' + v'v''] = 0$$

$$s_{12} = x'x'' + y'y'' + z'z'' + v'v'' = 0$$

$$s_{12}' = \left\{ \begin{array}{l} x'x''' + y'y''' + z'z''' + v'v''' + \\ x''x'' + y''y'' + z''z'' + v''v'' \end{array} \right\}$$

$$0 = s_{12}' = s_{13} + s_{22} = s_{13} + \frac{1}{\rho^2}$$

$$\boxed{s_{13} = -\frac{1}{\rho^2}} \qquad \boxed{G = -\frac{1}{\rho^2}}$$

$$s_{22} = x''x'' + y''y'' + z''z'' + v''v'' = \frac{1}{\rho^2}$$

$$s_{22}' = 2\left[x''x''' + y''y''' + z''z''' + v''v'''\right] = 2s_{23}$$

$$-\frac{2\rho'}{\rho^3} = 2s_{23}$$

$$\boxed{s_{23} = -\frac{\rho'}{\rho^3}}$$

$$\boxed{F = -\frac{\rho'}{\rho^3}}$$

$$s_{13} = -\frac{1}{\rho^2} = x'x''' + y'y''' + z'z''' + v'v'''$$

$$s_{13}' = +\frac{2\rho'}{\rho^3} = \left\{\begin{array}{l} x'x'''' + y'y^{IV} + z'z^{IV} + v'v^{IV} \\ + x''x''' + y''y''' + z''z''' + v''v''' \end{array}\right\}$$

$$\frac{2\rho'}{\rho^3} = s_{14} + s_{23}$$

$$\frac{2\rho'}{\rho^3} = s_{14} - \frac{\rho'}{\rho^3}$$

$$s_{14} = \frac{3\rho'}{\rho^3}$$

$$\boxed{L = s_{14} = \frac{3\rho'}{\rho^3}}$$

Definición: $C = s_{33} = x'''^2 + y'''^2 + z'''^2 + v'''^2$.

$$\boxed{C = \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} = \frac{R^2+\sigma^2}{\sigma^2\rho^4}}$$

$$\boxed{R^2 = \rho^2 + \sigma^2\rho'^2}$$

$$s_{23} = -\frac{\rho'}{\rho^3} = x''x''' + y''y''' + z''z''' + v''v'''.$$

$$s_{23}' = +\frac{3\rho'^2}{\rho^4} - \frac{\rho''}{\rho^3} = \left\{\begin{array}{l} x''x^{IV} + y''y^{IV} + z''z^{IV} + v''v^{IV} \\ + x'''x''' + y'''y''' + z'''z''' + v'''v''' \end{array}\right\}$$

$$\frac{3\rho'^2}{\rho^4} - \frac{\rho''}{\rho^3} = s_{24} + \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}$$

$$\boxed{s_{24} = \frac{2\rho'^2}{\rho^4} - \frac{\rho''}{\rho^3} - \frac{1}{\sigma^2\rho^2} - \frac{1}{\rho^4} = M}$$

$$s_{33} = x'''^2 + y'''^2 + z'''^2 + v'''^2;$$

$$s_{33}' = 2\left(x'''x^{IV} + y'''y^{IV} + z'''z^{IV} + v'''v^{IV}\right);$$

$$-\frac{2\sigma'}{\sigma^3\rho^2} - \frac{2\rho'}{\sigma^2\rho^3} - \underline{\frac{4\rho'}{\rho^5}} + \underline{\frac{2\rho'\rho''}{\rho^4}} - \underline{\frac{4\rho'^3}{\rho^5}} = 2s_{34}$$

$$\boxed{s_{34} = N = \frac{\rho'\rho''}{\rho^4} - \frac{2\rho'^3}{\rho^5} - \frac{2\rho'}{\rho^5} - \frac{\rho'}{\sigma^2\rho^3} - \frac{\sigma'}{\sigma^3\rho^2}}$$

Demostrar que $N+\frac{\rho'}{\rho}M+\frac{1}{\rho^2}L=\frac{-1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)$

$$N=\frac{\rho'\rho''}{\rho^4}-\frac{2\rho'^3}{\rho^5}-\frac{2\rho'}{\rho^5}-\frac{\rho'}{\sigma^2\rho^3}-\frac{\sigma'}{\sigma^3\rho^2}$$

$$\frac{\rho'}{\rho}M=-\frac{\rho'\rho''}{\rho^4}+\frac{2\rho'^3}{\rho^5}-\frac{\rho'}{\rho^5}-\frac{\rho'}{\sigma^2\rho^3}$$

$$\frac{1}{\rho^2}L= \qquad +\frac{3\rho'}{\rho^5}$$

$$N+\frac{\rho'}{\rho}M+\frac{1}{\rho^2}L=\frac{-1}{\sigma^2\rho^2}\left[2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right]$$

$$\boxed{N+\frac{\rho'}{\rho}M+\frac{1}{\rho^2}L=-\frac{1}{\sigma^2\rho^2}\left[2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right]}$$

$$\Omega = \begin{vmatrix} x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \\ x''' & y''' & z''' & v''' \\ x^{IV} & y^{IV} & z^{IV} & v^{IV} \end{vmatrix}$$

$$\Omega^2 = \begin{vmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{vmatrix}$$

$$\Omega^2 = \begin{vmatrix} 1 & 0 & -\frac{1}{\rho^2} & L \\ 0 & \frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} & M \\ -\frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} & \frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} & N \\ L & M & N & D \end{vmatrix}$$

$$\Omega^2 = \begin{vmatrix} 1 & 0 & -\frac{1}{\rho^2} & L \\ 0 & \frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} & M \\ 0 & -\frac{\rho'}{\rho^3} & \frac{1}{\sigma^2\rho^2} + \frac{\rho'^2}{\rho^4} & \frac{L}{\rho^2} + N \\ 0 & M & N + \frac{L}{\rho^2} & D - L^2 \end{vmatrix}$$

$$\Omega^2 = \begin{vmatrix} \frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} & M \\ 0 & \frac{1}{\sigma^2\rho^2} & \frac{L}{\rho^2} + N + \frac{M\rho'}{\rho} \\ 0 & N + \frac{L}{\rho^2} + \frac{M\rho'}{\rho} & D - L^2 - \rho^2 M^2 \end{vmatrix}$$

$$\frac{D}{\sigma^2\rho^2} - \frac{L^2}{\sigma^2\rho^2} - \frac{M}{\sigma^2} - \frac{LN}{\rho^2} - \frac{L^2}{\rho^4} - \frac{LM\rho'}{\rho^3}$$

$$- N^2 - LN$$

$$\frac{D}{\sigma^2\rho^4} - \frac{L^2}{\sigma^2\rho^4} - \frac{M^2}{\rho^2\sigma^2} - \frac{1}{\rho^2}\left(\frac{L}{\rho^2} + N + \frac{M\rho'}{\rho}\right)^2 = \Omega^2$$

$$\cancel{\Omega^2 = \frac{D}{\rho^2\sigma^2} -}$$

$$\boxed{\Omega^2 = \frac{D}{\rho^4\sigma^2} - \frac{M^2}{\rho^2\sigma^2} - \frac{L^2}{\rho^4\sigma^2} - \frac{1}{\rho^2}\left(N + \frac{\rho'}{\rho}M + \frac{1}{\rho^2}L\right)^2}$$

$$\boxed{\Omega^2 = \frac{D}{\rho^4\sigma^2} - \frac{M^2}{\rho^2\sigma^2} - \frac{L^2}{\rho^4\sigma^2} - \frac{1}{\rho^6\sigma^4}\left[2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right]^2}$$

Recta tangente y Llano normal.

Punto P: x, y, z, v

Punto P': $x+\Delta x, y+\Delta y, z+\Delta z, v+\Delta v$

Punto: $\bar{x}, \bar{y}, \bar{z}, \bar{v}$ en la recta tangente.

Ecuaciones de la recta tangente:

$$\frac{\bar{x}-x}{\Delta x} = \frac{\bar{y}-y}{\Delta y} = \frac{\bar{z}-z}{\Delta z} = \cancel{\frac{\overline{P}P}{\Delta s}} \frac{\bar{v}-v}{\Delta v} = \frac{\overline{P}P}{\Delta s}$$

$$\frac{\bar{x}-x}{\left(\frac{\Delta x}{\Delta s}\right)} = \frac{\bar{y}-y}{\left(\frac{\Delta y}{\Delta s}\right)} = \frac{\bar{z}-z}{\left(\frac{\Delta z}{\Delta s}\right)} = \frac{\bar{v}-v}{\left(\frac{\Delta v}{\Delta s}\right)} = \overline{P}P$$

$\lim \Delta s \to 0$

$$\boxed{\frac{\bar{x}-x}{x'} = \frac{\bar{y}-y}{y'} = \frac{\bar{z}-z}{z'} = \frac{\bar{v}-v}{v'} = \overline{P}P}$$ Recta tangente

Llano normal:

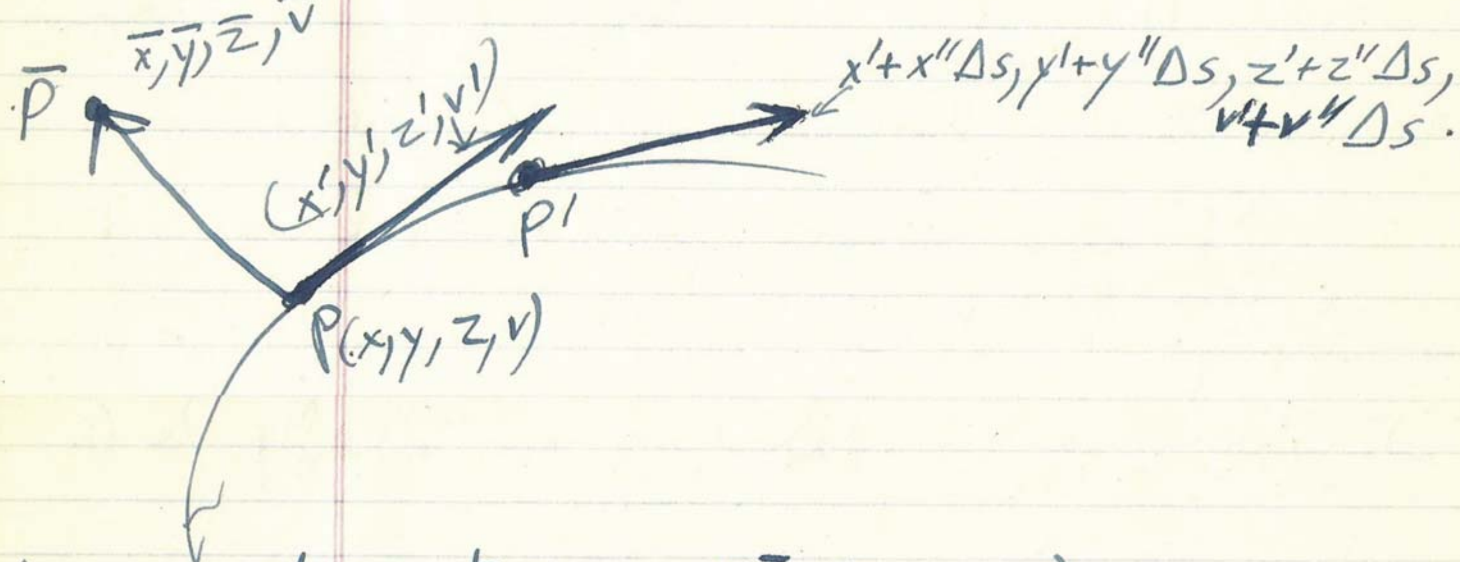
$$(\bar{x}-x)x' + (\bar{y}-y)y' + (\bar{z}-z)z' + (\bar{v}-v)v' = 0$$

Si $(\alpha, \beta, \gamma, \delta)$ son los cosenos directores de cualquier dirección en el llano

$$\Rightarrow \alpha x' + \beta y' + \gamma z' + \delta v' = 0$$

Plano Osculador.

Plano que contiene a la tangente en P y que es paralelo a la tangente en P'



Los cuadrivectores: $P\bar{P}$, (x', y', z', v') y $(x'+x''\Delta s, y'+y''\Delta s, \ldots$ son coplanos.

$(\bar{x}-x, \bar{y}-y, \bar{z}-z, \bar{v}-v)$ es una combinación lineal de (x', y', z', v') y $(x'+x''\Delta s, y'+y''\Delta s, \ldots)$

$$\begin{Vmatrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'+x''\Delta s & y'+y''\Delta s & z'+z''\Delta s & v'+v''\Delta s \end{Vmatrix} = 0$$

$$\begin{Vmatrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \end{Vmatrix} = 0$$

~~Plano osculador es el plano límite que cumple las siguientes condiciones~~

Plano osculador es el plano límite cuando P' tiende a P del plano que cumple las siguientes condiciones:

1) el plano está apoyado en P.
2) el plano contiene a la tangente en P
3) el plano es paralelo a la tangente en P'.

El plano osculador también se puede obtener como límite de un plano apoyado en P, P' y P'' cuando P' y P'' tienden a P simultáneamente y con independencia uno de otro.

x, y, z, v —— coordenadas de P

$x + x'\Delta s + \frac{1}{2}x''\Delta s^2, \cdots$ coordenadas de P'

$x + x'\Delta\sigma + \frac{1}{2}x'' + \Delta\sigma^2, \cdots$ coordenadas de P''

$\overline{P}(\bar{x}, \bar{y}, \bar{z}, \bar{v})$ punto corriente del plano osculador

Cuadrivectores $\bar{P}P$, $\bar{P}P'$, $\bar{P}P''$ son coplanos.

$$\cancel{\bar{x}-x \quad \bar{y}-y \quad \bar{z}-z \quad \bar{v}-v}$$

$$\cancel{\bar{x}-x-x'\Delta s-\tfrac{1}{2}x''}$$

$$0=\left\| \begin{array}{lll} \bar{x}-x & \bar{y}-y & \cdots \\ \bar{x}-x-x'\Delta s-\frac{1}{2}x''\Delta s^2 & \bar{y}-y-y'\Delta s-\frac{1}{2}y''\Delta s^2 & - \\ \bar{x}-x-x'\Delta\sigma-\frac{1}{2}x''\Delta\sigma^2 & \bar{y}-y-y'\Delta\sigma-\frac{1}{2}y''\Delta\sigma^2 & - \end{array} \right.$$

$$0=\left\| \begin{array}{ll} \bar{x}-x & - - - \\ -x'\Delta s-\frac{1}{2}x''\Delta s^2 & - - - \\ -x'\Delta\sigma-\frac{1}{2}x''\Delta\sigma^2 & - - - \end{array} \right\|$$

$$0=\left\| \begin{array}{ll} \bar{x}-x & - - \\ +x'+\frac{1}{2}x''\Delta s & - - \\ +x'+\frac{1}{2}x''\Delta\sigma & - - \end{array} \right\|$$

$$0=\left\| \begin{array}{ll} \bar{x}-x & - - \\ x'+\frac{1}{2}x''\Delta s & - - \\ \frac{1}{2}(x'')(\Delta\sigma-\Delta s) & - - \end{array} \right\|$$

$$0=\left\| \begin{array}{ll} \bar{x}-x & - - \\ x' & - - \\ x'' & - \end{array} \right\|$$

<u>El plano osculador contiene al cuadrivector velocidad y al cuadrivector aceleración.</u>

Considérese el llano normal en P.

$$(\bar{x}-x)x' + (\bar{y}-y)y' + (\bar{z}-z)z' + (\bar{v}-v)v' = 0$$

Considérese el llano normal en el punto P'

$$P'(x + x'\Delta s, y + y'\Delta s, z + z'\Delta s, v + v'\Delta s)$$

$$\cancel{(\bar{x}-x-x'\Delta s, \bar{y}-y-y'\Delta s, \bar{z}-z-z'\Delta s, \bar{v}-v-v'\Delta s)}$$

$$\sum (\bar{x}-x-x'\Delta s)(x'+x''\Delta s) = 0$$

$$\sum (\bar{x}-x)x' + \Delta s \sum\left[(\bar{x}-x)x'' - x'^2\right] - \Delta s^2 \sum x'x'' = 0$$

Intersección de los dos llanos

$$\sum (\bar{x}-x)x' = 0$$

$$\sum (\bar{x}-x)x' + \Delta s \sum\left[(\bar{x}-x)x'' - x'^2\right] = 0$$

es $\rightarrow \sum (\bar{x}-x)x'' = 1$

Plano definido por las <u>dos</u> ecuaciones

$$\sum (\bar{x}-x)x' = 0$$

$$\sum (\bar{x}-x)x'' = 1$$

El plano definido por las dos ecuaciones:

$$\sum (\bar{x}-x)\, x' = 0, \text{ y}$$

$$\sum (\bar{x}-x)\, x'' = 1;$$

es el plano límite de la intersección del llano normal en P y del llano normal en P', cuando $P' \to P$.

Todo cuadrivector contenido en ese plano es $\perp$ al cuadrivector tangente.

Sea $x_{(1)}, y_{(1)}, z_{(1)}, v_{(1)}$ un cuadrivector de posición de un punto particular del plano definido por las dos ecuaciones. $\Rightarrow$

$$\sum (\bar{x}-x)\, x' = 0; \quad \sum (x_{(1)}-x)\, x' = 0;$$

$$\sum (\bar{x}-x)\, x'' = 1; \quad \sum (x_{(1)}-x)\, x'' = 1.$$

$$\therefore \sum (\bar{x}-x_{(1)})\, x' = 0;$$

$$\sum (\bar{x}-x_{(1)})\, x'' = 0.$$

Todo cuadrivector $(\bar{x}-x_{(1)})$ contenido en el plano es perpendicular a todo cuadrivector contenido en el plano osculador.

$$\sum(\bar{x}-x_{(1)})(\alpha x'+\beta x'')=0$$

$\alpha x'+\beta x''$ es un cuadrivector cualquiera del plano osculador.

¡¡El plano ortogonal (al osculador) no pasa por el punto P.!!

Encuéntrese la intersección del plano osculador y del plano ortogonal.

Cuadrivector de posición de un punto en el plano osculador:

$$\bar{x}=x+px'+qx'', \dots$$

Determínense p y q de manera que este punto esté también en el plano ortogonal.

$$\sum(px'+qx'')x'=0;$$

$$\sum(px'+qx'')x''=1.$$

$$\sum px'x'+\sum qx'x''=0;$$

$$\sum px'x''+\sum qx''x''=1.$$

$$p + q s_{12} = 0 \;; \quad s_{12} = 0 \implies$$

$$p = 0$$

$$q s_{22} = 1. \qquad q \frac{1}{\rho^2} = 1$$

$$q = \rho^2$$

Punto de intersección del plano osculador y del plano ortogonal: **C**

$$x + \rho^2 x'', \; y + \rho^2 y'', \; z + \rho^2 z'', \; v + \rho^2 v''$$

Magnitud del cuadrivector PC.

$$|PC| = \sqrt{\rho^4 (x''^2 + y''^2 + z''^2 + v''^2)} = \sqrt{\rho^2} = \rho$$

Cosenos directores de PC (de P a C)

son: $\rho x''$, $\rho y''$, $\rho z''$, $\rho v''$.

Problema: ¿Existe un plano tal que todas las direcciones en él sean perpendiculares a todas las de un plano dado?. (Planos apoyados en el origen)

Plano dado Π: $x^i = a^i\lambda + b^i\mu$

λ y μ parámetros arbitrarios; a^i y b^i cuadrivectores constantes.

Plano ortogonal: $x^i = A^i\varphi + B^i\psi$

φ, ψ parámetros arbitrarios; A^i y B^i cuadrivectores fijos.

$$\sum_i (a^i\lambda + b^i\mu)(A^i\varphi + B^i\psi) = 0$$

para toda $\lambda, \mu, \varphi, \psi$

$$\sum_i \left[a^iA^i\lambda\varphi + a^iB^i\lambda\psi + b^iA^i\mu\varphi + b^iB^i\mu\psi\right] = 0$$

$\sum a^i A^i = 0$;
$\sum a^i B^i = 0$;
$\sum b^i A^i = 0$;
$\sum b^i B^i = 0$.

$$a^1A^1 + a^2A^2 + a^3A^3 + a^4A^4 = 0$$
$$b^1A^1 + b^2A^2 + b^3A^3 + b^4A^4 = 0$$

$$a^1B^1 + a^2B^2 + a^3B^3 + a^4B^4 = 0$$
$$b^1B^1 + b^2B^2 + b^3B^3 + b^4B^4 = 0$$

Solución: constrúyanse dos cuadrivectores A^i y B^i perpendiculares a los cuadrivectores a^i y b^i.

$$A^i = \left\| \begin{matrix} m^1 & m^2 & m^3 & m^4 \\ a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \end{matrix} \right\| ;$$

$$B^i = \left\| \begin{matrix} n^1 & n^2 & n^3 & n^4 \\ a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \end{matrix} \right\| .$$

$$\sum A^i a^i = 0 \; ; \quad \sum A^i b^i = 0 \; ;$$

$$\sum B^i a^i = 0 \; ; \quad \sum B^i b^i = 0 .$$

Condición $\begin{vmatrix} m^1 & m^2 & m^3 & m^4 \\ n^1 & n^2 & n^3 & n^4 \\ a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \end{vmatrix} \neq 0$.

Todo cuadrivector $A^i\varphi + B^i\psi$ es $\perp$ a todo cuadrivector $a^i\lambda + b^i\mu$.

Intersección de dos planos ortogonales:

$$c^i + a^i\lambda + b^i\mu = A^i\varphi + B^i\psi + C^i$$

$$\begin{vmatrix} a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \\ A^1 & A^2 & A^3 & A^4 \\ B^1 & B^2 & B^3 & B^4 \end{vmatrix} \neq 0$$

$a_E \quad a_i$

$$\begin{vmatrix} a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \\ A^1 & A^2 & A^3 & A^4 \\ B^1 & B^2 & B^3 & B^4 \end{vmatrix} \begin{vmatrix} a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \\ A^1 & A^2 & A^3 & A^4 \\ B^1 & B^2 & B^3 & B^4 \end{vmatrix} = \begin{vmatrix} \hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} & 0 & 0 \\ \hat{a}\cdot\hat{b} & \hat{b}\cdot\hat{b} & 0 & 0 \\ 0 & 0 & \hat{A}\cdot\hat{A} & \hat{A}\cdot\hat{B} \\ 0 & 0 & \hat{B}\cdot\hat{A} & \hat{B}\cdot\hat{B} \end{vmatrix}$$

$$= \left[(\hat{a}\cdot\hat{a})(\hat{b}\cdot\hat{b}) - (\hat{a}\cdot\hat{b})^2\right]\left[(\hat{A}\cdot\hat{A})(\hat{B}\cdot\hat{B}) - (\hat{A}\cdot\hat{B})^2\right]$$

Considérense tres puntos:

$P: \quad x, y, z, v$

$P': \quad x + x'\Delta s + \frac{1}{2}x''\Delta s^2, \ldots -$

$P'': \quad x + x'\Delta\sigma + \frac{1}{2}x''\Delta\sigma^2 + \ldots$

Agréguese el centro de curvatura C

$C: \quad x + \rho^2 x'', \; y + \rho^2 y'', \; z + \rho^2 z'', \; v + \rho^2 v''$

$PP': \quad x'\Delta s + \frac{1}{2}x''\Delta s^2, \ldots -$

$PC: \quad \rho^2 x'', \ldots -$

$$\widehat{PP'} \cdot \widehat{PC} = \sum\left[\rho^2 \Delta s\, x' x'' + \frac{1}{2}\rho^2 \Delta s^2\, x''^2\right] = \frac{1}{2}\Delta s^2$$

2

$$|\widehat{p'p''}|^2 \doteq (\sigma - s)^2 + \frac{(\sigma^2 - s^2)^2}{4\rho^2}$$

$$|\widehat{p'p''}| \doteq (\sigma - s)\left[1 + \frac{(\sigma + s)^2}{8\rho^2}\right]$$

$$|\widehat{p'p''}| \doteq (\sigma - s) + \frac{(\sigma - s)(\sigma + s)^2}{8\rho^2}$$

$$|\widehat{p'c}|^2 = s^2 + \frac{(\frac{1}{2}s^2 - \rho^2)^2}{\rho^2}$$

$$|\widehat{p'c}|^2 = \rho^2 + \frac{s^4}{4\rho^2}$$

$$|\widehat{p'c}| \doteq \rho\left[1 + \frac{s^4}{8\rho^4}\right] = \rho + \frac{s^4}{8\rho^3}$$

$$|p'c| = \rho + \frac{s^4}{8\rho^3}$$

$$\cancel{|\widehat{p'c}| \cdot |\widehat{p'p''}| = s^2 + \frac{s^2}{2\rho^2}\left(\tfrac{1}{2}s^2 - \rho^2\right)}$$

$$\cancel{|\widehat{p'c}| \cdot |\widehat{p'p''}|}$$

$$|\widehat{p'c}| \cdot |\widehat{p'p''}| = s(\sigma - s) + \frac{1}{2\rho^2}(\sigma^2 - s^2)\left(\tfrac{1}{2}s^2 - \rho^2\right)$$

$$|\widehat{p'c}| \cdot |\widehat{p'p''}| = s(\sigma - s) \cancel{+} - \tfrac{1}{2}(\sigma^2 - s^2) + \frac{s^2}{4\rho^2}(\sigma^2 - s^2)$$

$$|\widehat{p'c}| \cdot |\widehat{p'p''}| = s\sigma - s^2 - \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}s^2 + \frac{s^2}{4\rho^2}(\sigma^2 - s^2)$$

$$|\widehat{p'c}| \cdot |\widehat{p'p''}| = \cancel{-\tfrac{1}{2}(\sigma + \sigma)^2 \doteq \tfrac{1}{2}(s - \sigma}$$

$$|\widehat{p'c}| \cdot |\widehat{p'p''}| = -\tfrac{1}{2}(\sigma - s)^2 + \frac{s^2}{4\rho^2}(\sigma^2 - s^2)$$

Considérense los puntos:

$P: (x, y, z, v)$

$P': (x + x's + \frac{1}{2}x''s^2, y + y's + \frac{1}{2}y''s^2, \ldots)$

$P'': (x + x'\sigma + \frac{1}{2}x''\sigma^2, y + y'\sigma + \frac{1}{2}y''\sigma^2, \ldots)$

$C: (x + \rho^2 x'', y + \rho^2 y'', \ldots)$

Considérense los cuadrivectores:

$\widehat{PP'}, \widehat{P'P''}, \widehat{PC}$ y $\widehat{P'C}$.

$\widehat{PP'}: (x's + \frac{1}{2}x''s^2, y's + \frac{1}{2}y''s^2, \ldots)$

$\widehat{P'P''}: (x'\overline{\sigma - s} + \frac{1}{2}x''\overline{\sigma^2 - s^2}, y'\overline{\sigma - s} + \frac{1}{2}y''\overline{\sigma^2 - s^2}, \ldots)$

$\widehat{PC}: (\rho^2 x'', \rho^2 y'', \rho^2 z'', \ldots)$

$\widehat{P'C}: (x's + x''\overline{\tfrac{1}{2}s^2 - \rho^2}, y's + y''\overline{\tfrac{1}{2}s^2 - \rho^2}, \ldots)$

$|\widehat{PP'}|^2 = s^2 + \frac{1}{4}\frac{s^4}{\rho^2}$.

$|\widehat{PP'}| = s + \frac{s^3}{8\rho^2} \qquad |\widehat{PC}| = \rho$

$\widehat{PP'} \cdot \widehat{PC} = \frac{\rho^2 s^2}{2}\sum x''^2 = \frac{s^2}{2}$

$\left(s + \frac{s^3}{8\rho^2}\right)\rho \cos(\widehat{PP'}, \widehat{PC}) = \frac{s^2}{2}$

$\cos(\widehat{PP'}, \widehat{PC}) = \frac{s^2}{2\rho\left(s + \frac{s^3}{8\rho^2}\right)} = \frac{s}{2\rho\left(1 + \frac{s^2}{8\rho^2}\right)} \to 0$

$\widehat{PP'} \perp \widehat{PC}$

$$|\widehat{p'C}|\cdot|\widehat{p'p''}| = \left[\rho+\frac{s^4}{8\rho^3}\right]\left[(\sigma-s)+\frac{(\sigma-s)(\sigma+s)^2}{8\rho^2}\right]\cos\left[\widehat{p'C},\ \widehat{p'p''}\right]$$

$$|\widehat{p'C}|\cdot|\widehat{p'p''}| = (\sigma-s)\left[-\tfrac{1}{2}(\sigma-s)+\frac{s^2}{4\rho^2}(\sigma+s)\right]$$

$$\cos\left[\widehat{p'C},\ \widehat{p'p''}\right] = \frac{-\frac{1}{2}(\sigma-s)+\frac{s^2}{4\rho^2}(\sigma+s)}{\left[\rho+\frac{s^4}{8\rho^3}\right]\left[1+\frac{(\sigma+s)^2}{8\rho^2}\right]}$$

Normal Principal.

$$\frac{\bar{x}-x}{x''} = \frac{\bar{y}-y}{y''} = \frac{\bar{z}-z}{z''} = \frac{\bar{v}-v}{v''}$$

La normal principal está en el plano osculador

$$\left\|\begin{matrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \end{matrix}\right\| = 0$$

La normal principal está en el llano normal

$$(\bar{x}-x)x' + (\bar{y}-y)y' + (\bar{z}-z)z' + (\bar{v}-v)v' = 0$$

Angulo de Contingencia $d\varepsilon$.

La tangente en P: (x', y', z', v')

La tangente en P': $(x'+x''s, y'+y''s, z'+z''s, v'+v''s)$

(x', y', z', v') cuadrivector unitario

$$|(x'+x''s, y'+y''s, z'+z''s, v'+v''s)|^2 =$$

$$\cancel{1+\frac{s^2}{\rho^2}}$$

$$\cancel{|(x'+x''s, y'+y''s, \ldots)| \approx 1+\frac{s^2}{2\rho^2}}$$

$$\cancel{1\cdot\left(1+\frac{s^2}{2\rho^2}\right)\cos\varepsilon = 1}$$

tangente en P': $(x'+x''s+\frac{1}{2}x'''s^2, \ldots)$

$$|\text{tangente en } P'|^2 \approx 1+(s_{13}+s_{22})s^2$$

$$|\text{tangente en } P'|^2 \approx 1+\left(-\frac{1}{\rho^2}+\frac{1}{\rho^2}\right)s^2 = 1$$

$$1\cdot 1 \cos d\varepsilon = 1+\frac{1}{2}s_{13}s^2+\ldots$$

$$\boxed{\cos d\varepsilon = 1-\frac{s^2}{2\rho^2}}$$

$$1-\cos d\varepsilon = 2\,\mathrm{sen}^2\frac{d\varepsilon}{2}$$

$$2\,\mathrm{sen}^2\frac{d\varepsilon}{2} = \frac{s^2}{2\rho^2}$$

$$\boxed{\mathrm{sen}\frac{d\varepsilon}{2} = \frac{s}{2\rho}} \qquad d\varepsilon \approx \frac{s}{\rho}$$

$$\boxed{\frac{1}{\varrho} = \frac{d\varepsilon}{ds}}$$

Fórmula para el seno cuadrado del ángulo formado por dos cuadrivectores unitarios.

$(l, m, n, k) \quad (l', m', n', k')$ $\Big\{$ Cuadrivectores unitarios

Ángulo θ

$$\cos\theta = ll' + mm' + nn' + kk'$$

$\cancel{1 - \cos\theta}$

$$\text{sen}^2\theta = 1 - (ll' + mm' + nn' + kk')^2$$

$\cancel{\text{sen}^2\theta = 1 - 2(ll'}$

$$= \begin{cases} 1 - l^2l'^2 - m^2m'^2 - n^2n'^2 - k^2k'^2 \\ -2lml'm' - 2lnl'n' - 2lkl'k' \\ -2mnm'n' - 2mkm'k' - 2nkn'k' \end{cases}$$

$$(l^2 + m^2 + n^2 + k^2)(l'^2 + m'^2 + n'^2 + k'^2) = 1$$

$$1 = \begin{cases} l^2l'^2 + m^2m'^2 + n^2n'^2 + k^2k'^2 \\ + l^2m'^2 + l^2n'^2 + l^2k'^2 + m^2l'^2 + m^2n'^2 + \\ + m^2k'^2 + n^2l'^2 + n^2m'^2 + n^2k'^2 \\ + k^2l'^2 + k^2m'^2 + k^2n'^2 \end{cases}$$

$$\text{sen}^2\theta = \begin{cases} (lm' - ml')^2 + (ln' - nl')^2 \\ + (lk' - kl')^2 + (mn' - nm')^2 \\ + (mk' - km')^2 + (nk' - kn')^2 \end{cases}$$

$$\operatorname{sen}^2\theta = \sum (lm' - ml')^2$$

$$\left\| \begin{matrix} l & m & n & k \\ l' & m' & n' & k' \end{matrix} \right\|$$

$$\boxed{\operatorname{sen}^2\theta = \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}^2 + \begin{vmatrix} l & n \\ l' & n' \end{vmatrix}^2 + \begin{vmatrix} l & k \\ l' & k' \end{vmatrix}^2 + \begin{vmatrix} m & n \\ m' & n' \end{vmatrix}^2 + \begin{vmatrix} m & k \\ m' & k' \end{vmatrix}^2 + \begin{vmatrix} n & k \\ n' & k' \end{vmatrix}^2}$$

Angulo de contingencia:

tangente en P: x', y', z', v'

tangente en P': $x' + x''\Delta s, y' + y''\Delta s, \ldots$

$(\text{longitud de la tangente en } P)^2 = 1$

$(\text{longitud de la tangente en } P')^2 = 1 + \frac{\Delta s^2}{\rho^2}$

$$\operatorname{sen}^2\theta = \sum \frac{(x'y'' - x''y')^2 \Delta s^2}{\left(1 + \frac{\Delta s^2}{\rho^2}\right)^2}$$

$$\operatorname{sen}^2\theta = \left[\sum x'^2(x''^2 + y''^2 + z''^2 + v''^2) - \sum x'x''y'y''\right]\Delta s^2$$

$$\operatorname{sen}^2\theta = \left\{ \begin{matrix} (x'^2 + y'^2 + z'^2 + v'^2)(x''^2 + y''^2 + z''^2 + v''^2) \\ -\left[(x'x'' + y'y'' + z'z'' + v'v'')^2\right] \end{matrix} \right\} \Delta s^2$$

$$\operatorname{sen}^2\theta = \frac{\Delta s^2}{\rho^2} \qquad \boxed{\operatorname{sen}\theta = \frac{\Delta s}{\rho}}$$

$$\rho = \frac{ds}{d\varepsilon}$$

$$(x'y''-x''y')^2 = x'^2y''^2 + x''^2y'^2 - 2x'x''y'y''$$

$$(x'z''-x''z')^2 = x'z''^2 + x''^2z'^2 - 2x'x''z'z''$$

$$(x'v''-v''x')^2 = x'^2v''^2 + x''^2v'^2 - 2x'x''v'v''$$

$$(y'z''-y''z')^2 = y'^2z''^2 + y''^2z'^2 - 2y'y''z'z''$$

$$(y'v''-y''v')^2 = y'^2v''^2 + y''^2v'^2 - 2y'y''v'v''$$

$$(z'v''-z''v')^2 = z'^2v''^2 + z''^2v'^2 - 2z'z''v'v''$$

$$(x'x''-x''x')^2 = x'^2x''^2 + x''^2x'^2 - 2x'x''x'x''$$

$$(y'y''-y''y')^2 = y'^2y''^2 + y''^2y'^2 - 2y'y''y'y''$$

$$(z'z''-z''z')^2 = z'^2z''^2 + z''^2z'^2 - 2z'z''z'z''$$

$$(v'v''-v''v')^2 = v'^2v''^2 + v''^2v'^2 - 2v'v''v'v''$$

$$\left\{\left(\begin{matrix} x'^2 \\ +y'^2 \\ +z'^2 \\ +v'^2 \end{matrix}\right)(x''^2+y''^2+z''^2+v''^2)\right\} - \left\{\begin{matrix} x'x'' \\ +y'y'' \\ +z'z'' \\ +v'v'' \end{matrix}\right\}^2$$

$$= \frac{1}{\rho^2} - 0^2$$

Angulo que forman dos normales principales consecutivas.

Coeficientes directores de la normal en P $\Big\}$ x'', y'', z'', v''

$$\|(x'', y'', z'', v'')\| = \frac{1}{\rho}$$

Cosenos directores de la normal en P $\Big\}$ $\rho x'', \rho y'', \rho z'', \rho v''$

Normal vecina $\left(\rho x'' + \frac{d}{ds}(\rho x'')ds, \ \ldots\ldots \right.$

$$\operatorname{sen}^2 di = \sum \begin{vmatrix} \rho x'' & \rho y'' \\ \rho x'' + \frac{d}{ds}(\rho x'')ds & \rho y'' + \frac{d}{ds}(\rho y'')ds \end{vmatrix}^2$$

$$\operatorname{sen}^2 i \ \cancel{\phi} = \sum \begin{vmatrix} \rho x'' & \rho y'' \\ \cancel{\rho' x''} + \rho x''' & \cancel{\rho' y''} + \rho y''' \end{vmatrix}^2 ds^2$$

$$\operatorname{sen}^2 i = \sum \begin{vmatrix} \rho x'' & \rho y'' \\ \rho x''' & \rho y''' \end{vmatrix}^2 ds^2$$

$$\operatorname{sen}^2 i = \rho^4 ds^2 \left[(x'' y''' - x''' y'')^2 \right.$$

$$\operatorname{sen}^2 i = \rho^4 ds^2 \left(\sum x''^2 \sum x'''^2 - \left(\sum x'' x''' \right)^2 \right)$$

$$\operatorname{sen}^2 i = \rho^4 ds^2 \left[\frac{1}{\rho^2} S_{33} - S_{23}^2 \right]$$

$$\operatorname{sen}^2 i = \left[\rho^2 S_{33} - \rho^4 S_{23}^2 \right] ds^2$$

$$\operatorname{sen}^2 \iota = \left[\rho^2\left\{\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\right\} - \rho^4\left\{\frac{\rho'^2}{\rho^6}\right\}\right] d$$

$$\operatorname{sen}^2 \iota = \left[\frac{1}{\sigma^2} + \frac{1}{\rho^2} + \cancel{\frac{\rho'^2}{\rho^2}} - \cancel{\frac{\rho'^2}{\rho^2}}\right] ds^2$$

$$\operatorname{sen}^2 \iota = \left[\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right] ds^2$$

$$\operatorname{sen} \iota = \sqrt{\frac{1}{\sigma^2} + \frac{1}{\rho^2}}\, ds$$

$$\left(\frac{d\iota}{ds}\right)^2 = \frac{1}{\sigma^2} + \frac{1}{\rho^2}$$

$$\boxed{\left(\frac{d\varepsilon}{ds}\right)^2 = \frac{1}{\rho^2}}$$

$\dfrac{d\iota}{ds}$ es la curvatura de tornillo.

Dirección ~~en~~ arbitraria en el plano osculador.

Plano osculador: $\begin{Vmatrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \end{Vmatrix} = 0$

El plano osculador está definido por los cuadrivectores:

(x', y', z', v')
(x'', y'', z'', v'')

Cuadrivector de dirección arbitraria, <u>unitario</u>, en el plano osculador.

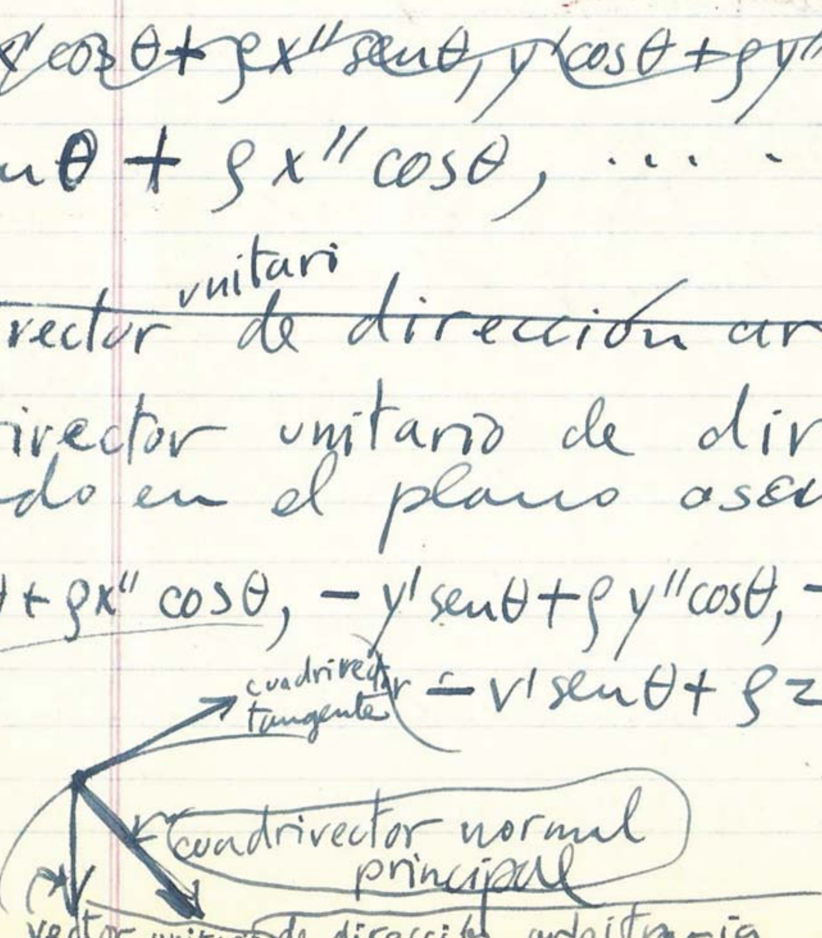
$\cancel{x'\cos\theta + \rho x''\operatorname{sen}\theta,\ y'\cos\theta + \rho y''\operatorname{sen}\theta,\ \ldots}$

$-x'\operatorname{sen}\theta + \rho x''\cos\theta,\ \ldots$

~~Cuadrivector unitari de dirección arbitrar~~

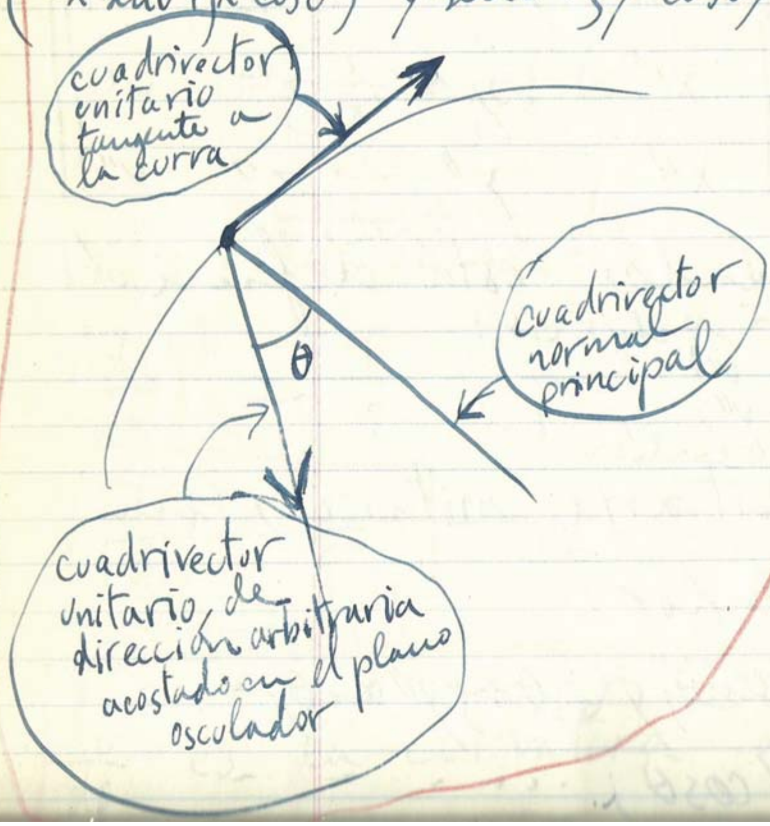
Cuadrivector unitario de dirección arbitraria colocado en el plano osculador.

$$\left(-x'\operatorname{sen}\theta + \rho x''\cos\theta,\ -y'\operatorname{sen}\theta + \rho y''\cos\theta,\ -z'\operatorname{sen}\theta + \rho z''\cos\theta,\ -v'\operatorname{sen}\theta + \rho z''\cos\theta\right)$$



Cuadrivector unitario de dirección arbitraria acostado en el plano osculador:

$$(-x'\operatorname{sen}\theta + \rho x''\cos\theta, -y'\operatorname{sen}\theta + \rho y''\cos\theta, -z'\operatorname{sen}\theta + \rho z''\cos\theta, -v'\operatorname{sen}\theta + \rho v''\cos\theta)$$

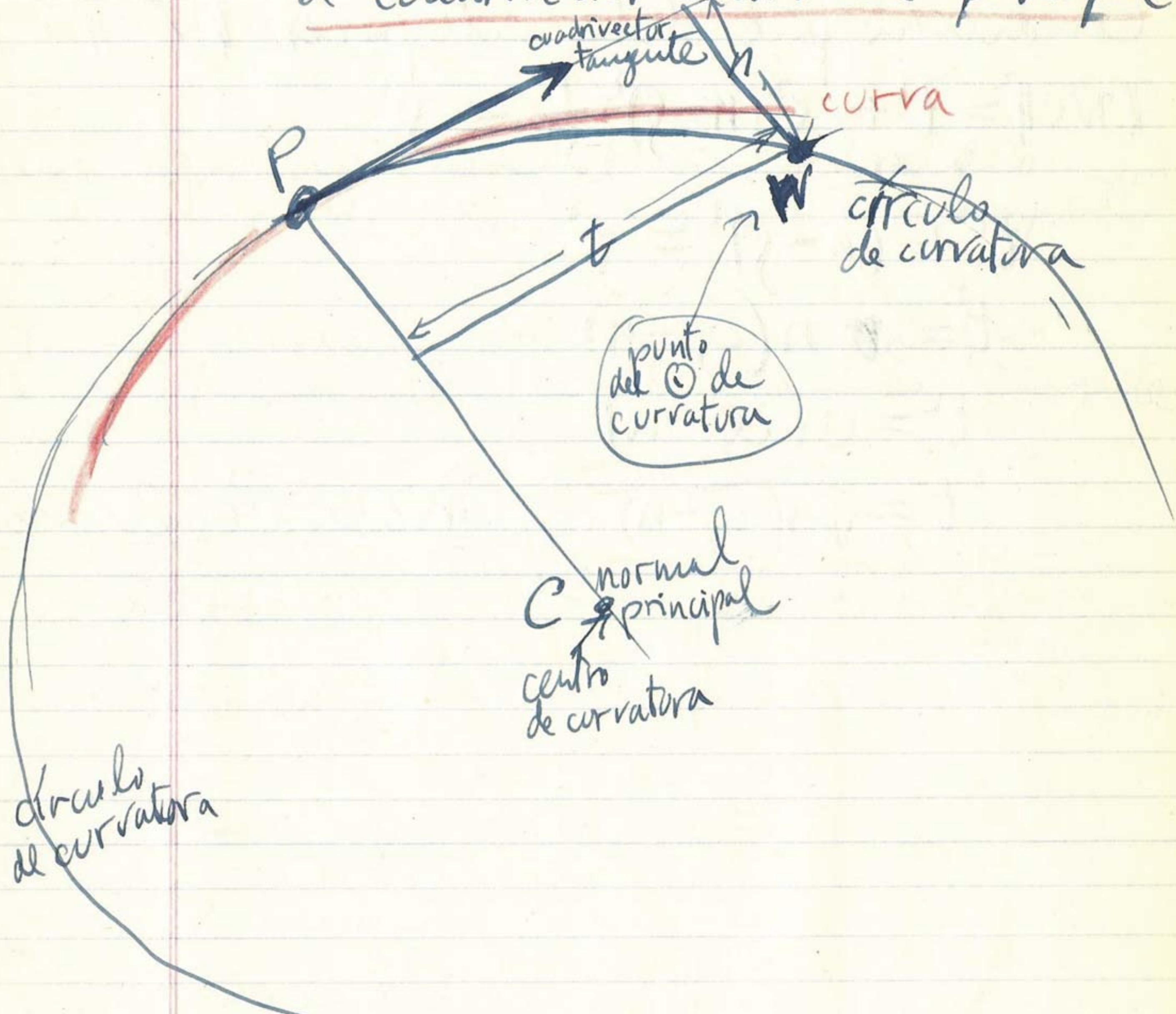


$\theta + \rho v'' \cos\theta)$

Ecuación del círculo de curvatura.

(En el plano osculador.)

Ejes de coordenadas:
el cuadrivector tangente y
el cuadrivector normal principal



Cuadrivector de posición de W:

$$x + tx' + n\rho x'',\ y + ty' + n\rho y'',\ z + tz' + n\rho z'',\ v + tv' + n\rho v''$$

Cuadrivector de posición de C :

$x+\rho^2 x''$, $y+\rho^2 y''$, $z+\rho^2 z''$, $v+\rho^2 v''$

Cuadrivector $\overline{WC}$

$tx'+(n\rho-\rho^2)x''$, $ty'+(n\rho-\rho^2)y''$, $tz'+(n\rho-\rho^2)z''$, $tv'+(n\rho-\rho^2)v''$

Condición que debe satisfacer $\|\overline{WC}\|$

$$\|\overline{WC}\|^2 = t^2 + \rho^2(n-\rho)^2 \cdot \frac{1}{\rho^2} = \rho^2$$

$$t^2 + (n-\rho)^2 = \rho^2$$

$$t^2 = n(2\rho - n)$$

$$t^2 = n(2\rho - n)$$

$$\boxed{t = \sqrt{n(2\rho - n)}}$$ ¡Círculo de Curvatura!

C

Π_1 - Pla

a

en

Π_2 —

P:

P':

Torsión: Angulo de Torsión:

Calcúlese el ángulo formado por dos planos osculadores vecinos.

$$\left\| \begin{matrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \end{matrix} \right\| = 0 \quad \text{Plano osculador}$$

Π_1 – Plano apoyado en P y paralelo a la tangente en P y a la tangente en P'.

Π_2 — Plano apoyado en P' y paralelo a la tangente en P' y a la tangente en P''.

		tangente
P:	$x,\ y,\ z,\ v$	$x', \ldots$
P':	$x + x's + \frac{1}{2}x''s^2, \ldots$	$x' + x''s + \frac{1}{2}x'''s^2, \ldots$
P'':	$x + x'\sigma + \frac{1}{2}x''\sigma^2, \ldots$	$x' + x''\sigma + \frac{1}{2}x'''\sigma^2, \ldots$

$$\Pi_1: \left\| \begin{matrix} \cdot & \bar{x}-x & - & - \\ \cdot & x' & - & - \\ & x' + x''s + \frac{1}{2}x'''s^2 & - & \end{matrix} \right\| = 0$$

$$\Pi_1 : \quad \begin{Vmatrix} \bar{x} - x & — \\ x' & — \\ x' + x''s + \frac{1}{2}x'''s^2 & — \end{Vmatrix} = 0$$

$$\Pi_2 : \quad \begin{Vmatrix} \bar{x} - x - x's - \frac{1}{2}x''s^2 & — \\ x' + x''s + \frac{1}{2}x'''s^2 & — \\ x' + x''\sigma + \frac{1}{2}x'''\sigma^2 & — \end{Vmatrix} = 0$$

$$\Pi_1 : \quad \begin{Vmatrix} \bar{x} - x & — \\ x' \cancel{+ x''s + \frac{1}{2}x'''s} & — \\ x''s + \frac{1}{2}x'''s^2 & — \end{Vmatrix} = 0$$

$$\Pi_2 : \quad \begin{Vmatrix} \bar{x} - x - x's - \frac{1}{2}x''s^2 & — \\ x' + x''s + \frac{1}{2}x'''s^2 & — \\ x''(\sigma - s) + \frac{1}{2}x'''(\sigma^2 - s^2) & — \end{Vmatrix} = 0$$

Coeficientes directores de una $\perp$ a la recta:

$$x' + x''s + \frac{1}{2}x'''s^2, \; — — —$$

en el plano Π_1

$$\alpha x' + \beta(x' + x''s + \tfrac{1}{2}x'''s^2), \ldots\ldots$$

$$\perp \quad \alpha \quad x' + x''s + \tfrac{1}{2}x'''s^2, \ldots\ldots$$

$$\alpha + \cancel{\alpha s_{12}} s + \tfrac{1}{2}\alpha s_{13} s^2 + \beta + \cancel{\beta s_{12} s} + \tfrac{3}{2}\beta s_{13} s^2 + \beta s_{22} s^2$$

$$\alpha = -\beta$$

$$-\alpha x' + \alpha(x' + x''s + \tfrac{1}{2}x'''s^2)$$

$$\gamma(x' + x''s + \tfrac{1}{2}x'''s^2) + \delta(x' + x''\sigma + \tfrac{1}{2}x'''\sigma^2)$$

$$\perp \quad (x' + x''s + \tfrac{1}{2}x'''s^2)$$

$$\cancel{\gamma} + \cancel{\tfrac{1}{2}\gamma s_{13} s^2} + \tfrac{1}{2}\gamma s_{13} s^2 + \gamma s_{22} s^2 + \cancel{\delta} + \cancel{\tfrac{1}{2}\delta s_{13} s^2}$$

$$+ \tfrac{1}{2}\delta s_{13}\sigma^2 + \delta s_{22} s\sigma$$

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$$\Pi_1 : \left\| \begin{matrix} \bar{x} - x & - & - \\ x' & - & - \\ x'' + \frac{1}{2} x''' s & - & - \end{matrix} \right\| = 0$$

$$\Pi_2 : \left\| \begin{matrix} \bar{x} - x - x' s - \frac{1}{2} x'' s^2 & - \\ x' + x'' s + \frac{1}{2} x''' s^2 & - \quad - \\ x'' + \frac{1}{2} x''' (\sigma + s) & - - - \end{matrix} \right\| = 0$$

$$\alpha x' + \beta (x'' + \tfrac{1}{2} x''' s) \quad - \quad -$$

$$\perp \quad x' + x'' s + \tfrac{1}{2} x''' s^2 \quad - \quad -$$

$$\cancel{\alpha + \frac{\beta}{\varrho^2} s = 0 \qquad \alpha = 0}$$

$$\cancel{\beta = -2 \frac{\alpha \varrho^2}{s}}$$

Angulo que forman dos planos que tienen una ~~recta~~ dirección común.

$$\left\|\begin{matrix} x-x_1 & y-y_1 & z-z_1 & v-v_1 \\ a & b & c & d \\ \alpha & \beta & \gamma & \delta \end{matrix}\right\| = 0 \qquad \pi_1$$

$$\left\|\begin{matrix} x-x_1 & y-y_1 & z-z_1 & v-v_1 \\ a & b & c & d \\ r & s & t & u \end{matrix}\right\| = 0 \qquad \pi_2$$

Dirección $\perp$ a (a, b, c, d) en π_1:

$$(\lambda a+\mu\alpha,\ \lambda b+\mu\beta,\ \lambda c+\mu\gamma,\ \lambda d+\mu\delta)$$

$$(\ a\ ,\ b,\ c,\ d\ )$$

$$\lambda(a^2+b^2+c^2+d^2)+\mu(a\alpha+b\beta+c\gamma+d\delta)=0$$

$$\boxed{\begin{aligned}\lambda &= (a\alpha+b\beta+c\gamma+d\delta)k\\ \mu &= (a^2+b^2+c^2+d^2)k\end{aligned}}$$

Cuadrivector unitario $\perp$ a (a,b,c,d) en π_1

$\cancel{k^2 a^2+b^2+c^2+d^2}$

$$\left.\begin{aligned} &k^2(a\alpha+b\beta+c\gamma+d\delta)^2(a^2+b^2+c^2+d^2)\\ &+k^2(a\alpha+b\beta+c\gamma+d\delta)^2(a^2+b^2+c^2+d^2)\\ &+k^2(a^2+b^2+c^2+d^2)^2(\alpha^2+\beta^2+\gamma^2+\delta^2)\end{aligned}\right\} = 1$$

$$k^2\left[2(a\alpha+b\beta+c\gamma+d\delta)^2(a^2+b^2+c^2+d^2)+(a^2+b^2+c^2+d^2)^2(\alpha^2+\beta^2+\gamma^2+\delta^2)\right]=1$$

$$k=\frac{1}{\sqrt{2(a\alpha+b\beta+c\gamma+d\delta)^2(a^2+b^2+c^2+d^2)+(a^2+b^2+c^2+d^2)^2(\alpha^2+\beta^2+\gamma^2+\delta^2)}}$$

$$k=\frac{1}{\sqrt{(a^2+b^2+c^2+d^2)\left[2(a\alpha+b\beta+c\gamma+d\delta)^2+(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)\right]}}$$

$$\lambda=\frac{a\alpha+b\beta+c\gamma+d\delta}{\sqrt{(a^2+b^2+c^2+d^2)\left[2(a\alpha+b\beta+c\gamma+d\delta)^2+(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)\right]}}$$

$$\mu=\frac{a^2+b^2+c^2+d^2}{\sqrt{(a^2+b^2+c^2+d^2)\left[2(a\alpha+b\beta+c\gamma+d\delta)^2+(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)\right]}}$$

$(\lambda a+\mu\alpha,\ \lambda b+\mu\beta,\ \lambda c+\mu\gamma,\ \lambda d+\mu\delta)^*$

es $\perp$ a a, b, c, d y está en el plano π_{ij}; es además unitario.

$\nu = \frac{}{\sqrt{(a^{\ldots}}}$

$\varphi = \frac{}{\sqrt{(a\ldots}}$

$(\nu a+\varphi\ldots$

es $\perp$ a

es adem…

$$\gamma = \frac{ar+bs+ct+du}{\sqrt{(a^2+b^2+c^2+d^2)\left[2(ar+bs+ct+du)^2+(a^2+b^2+c^2+d^2)(r^2+s^2+t^2+u^2)\right]}}$$

$$\varphi = \frac{r^2+s^2+t^2+u^2}{\sqrt{(a^2+b^2+c^2+d^2)\left[2(ar+bs+ct+du)^2+(a^2+b^2+c^2+d^2)(r^2+s^2+t^2+u^2)\right]}}$$

$(\gamma a+\varphi r,\ \gamma b+\varphi s,\ \gamma c+\varphi t,\ \gamma d+\varphi u)$

es $\perp$ a (a,b,c,d) y está en el plano π_2;

es además unitario.

Seno^2 del ángulo que forman los dos planos.

$$\operatorname{sen}^2 \varepsilon = \sum \begin{vmatrix} \lambda a + \mu\alpha & \lambda b + \mu\beta \\ \nu a + \varphi r & \nu b + \varphi s \end{vmatrix}^2$$

$$\begin{vmatrix} \lambda a + \mu\alpha & \lambda b + \mu\beta \\ \nu a + \varphi r & \nu b + \varphi s \end{vmatrix} = \begin{cases} \cancel{\lambda\nu(ab - ab)} \\ \cancel{+\lambda\varphi(as - br)} \\ \cancel{+\mu\nu} \end{cases}$$

$\cancel{\lambda a + \mu\alpha}$

$$\cancel{\begin{vmatrix} \lambda & \mu \\ \nu & \varphi \end{vmatrix} \begin{vmatrix} a & \beta \\ \alpha & s \end{vmatrix}}$$

$\cancel{\lambda\nu(ab - ab)}$

$\cdot\lambda\varphi(as - br)$

$\mu\nu(\alpha b - a\beta)$

$\mu\varphi(\alpha s - \beta r)$

$$\operatorname{sen}^2 \varepsilon = \begin{cases} \lambda^2\varphi^2 \sum (as - br)^2 \\ + \mu^2\nu^2 \sum (ab - a\beta)^2 \\ + \mu^2\varphi^2 \sum (\alpha s - \beta r)^2 \\ + 2\lambda\mu\varphi\nu \sum (as - br)(ab - a\beta) \end{cases}$$

Plano Π_1:

$$\left| \begin{matrix} \bar{x}-x & \text{---} \\ x' & \text{---} \\ x'+x''s+\frac{1}{2}x'''s^2 & \text{---} \end{matrix} \right\| = 0$$

$$\alpha x' + \beta(x'+x''s+\tfrac{1}{2}x'''s^2), \quad - - - - -$$

$$x'+x''s+\tfrac{1}{2}x'''s^2 \quad) \quad - - - -$$

$$\alpha(1+\tfrac{1}{2}s_{13}s^2) + \beta\left[1+\overline{s_{22}+s_{13}}\,s^2\right] = 0$$

;

$$\alpha = k\left[1+\overline{s_{22}+s_{13}}\,s^2\right]$$

$$\beta = -k\left[1+\tfrac{1}{2}s_{13}s^2\right]$$

$$\alpha^2 + \alpha\beta\left[2+ s_{13}s^2\right] + \beta^2\left[1+\overline{s_{22}+s_{13}}\,s^2\right] = 1$$

$$\left.\begin{matrix} k^2\left[1+2\,\overline{s_{22}+s_{13}}\,s^2\right] - k^2\left[1+\overline{s_{22}+\tfrac{3}{2}s_{23}}\,s^2\right]\left[2+s_{13}s^2\right] \\ + k^2\left[1+s_{13}s^2\right]\left(1+\overline{s_{22}+s_{13}}\,s^2\right) \end{matrix}\right\} = 1$$

$$k^2\left[\begin{matrix} 1+2\,\overline{s_{22}+s_{13}}\,s^2 + 2 + s_{13}s^2 + \overline{2s_{22}+3s_{23}}\,s^2 \\ +1+s_{13}s^2+\overline{s_{22}+s_{13}}\,s^2 \end{matrix}\right] = 1$$

$$k = \frac{1}{\sqrt{4+\overline{5s_{13}+5s_{22}+3s_{23}}\,s^2}}$$

$$k = \frac{1}{2}\left[1 - \left[\frac{5}{8}s_{13} + \frac{5}{8}s_{22} + \frac{3}{8}s_{23}\right]s^2\right]$$

Plano Π_1:

$$\begin{Vmatrix} \bar{x}-x & \bar{y}-y & \bar{z}-z & \bar{v}-v \\ x' & y' & z' & v' \\ x'+x''s+\frac{1}{2}x'''s^2 & - & - & - \end{Vmatrix} = 0$$

Dirección contenida en el plano Π_1:

$$\alpha x' + \beta\left(x'+x''s+\tfrac{1}{2}x'''s^2\right), \; - - - - -$$

$$(\alpha+\beta)x' +$$

Plano Π_1:

$$\begin{Vmatrix} \bar{x}-x & - & - \\ x' & - & - \\ x''+\frac{1}{2}x'''s & - & - \end{Vmatrix} = 0$$

Dirección arbitrario en Π_1:

$$\alpha x' + \beta\left(x'' + \tfrac{1}{2}x'''s\right), \quad \ldots$$

$$x' + x''s + \tfrac{1}{2}x'''s^2, \quad \ldots$$

$$\alpha + \tfrac{1}{2}\beta s_{13} s = 0 \qquad \alpha = -\tfrac{1}{2}\beta s_{13} s$$

$$\alpha^2 + \alpha\beta s_{13} s + \beta^2 s_{22} + \beta^2 s_{23} s = 1$$

$$\cancel{+\tfrac{1}{4}\beta^2 s_{13}^2 s^2}$$

$$\beta^2 = \frac{1}{s_{22} + s_{23} s}$$

$$\beta^2 = \frac{1}{\frac{1}{\rho^2} - \frac{\rho'}{\rho^3}s} = \frac{\rho^2}{1 - \frac{\rho'}{\rho}s} \approx \rho^2\left(1 + \frac{\rho'}{\rho}s\right)$$

$$\beta \approx \rho\left(1 + \frac{\rho'}{2\rho}s\right)$$

$$\alpha \approx +\frac{\rho}{2\rho}\left(1 + \frac{\rho'}{2\rho}s\right)$$

$$\boxed{\alpha \approx \frac{s}{2\rho}} \qquad \beta \approx \rho\left(1 + \frac{\rho'}{2\rho}s\right)$$

$$\boxed{\beta \approx \rho + \frac{\rho'}{2}s}$$

$$\frac{s}{2\rho}x' + \left(\rho + \frac{\rho'}{2}s\right)\left(x'' + \frac{1}{2}x'''s\right), \ldots$$

$$\cancel{x' + x''s + \frac{1}{2}x'''s^2}, \ldots$$

$$\cancel{\frac{s}{2\rho}} + \frac{s}{4\rho}\,\delta_{13}\,s^2 +$$

$$\Sigma\left[k\frac{s}{2\rho}x' + k\left(\rho + \frac{\rho'}{2}s\right)\left(x'' + \frac{1}{2}x'''s\right)\right]^2 = 1$$

$$\frac{k^2}{4\rho^2}$$

$$\left\| \begin{matrix} x-x_1 & y-y_1 & z-z_1 & v-v_1 \\ l_1 & m_1 & n_1 & k_1 \\ \lambda_1 & \mu_1 & \nu_1 & \kappa_1 \end{matrix} \right\| = 0 \quad \text{plano } \pi_1$$

Sextivector asociado al plano π_1

$$\left\| \begin{matrix} l_1 & m_1 & n_1 & k_1 \\ \lambda_1 & \mu_1 & \nu_1 & \kappa_1 \end{matrix} \right\|$$

$$\underset{12}{+\begin{vmatrix} l_1 & m_1 \\ \lambda_1 & \mu_1 \end{vmatrix}}, \quad \underset{13}{-\begin{vmatrix} l_1 & n_1 \\ \lambda_1 & \nu_1 \end{vmatrix}}, \quad \underset{14}{+\begin{vmatrix} l_1 & k_1 \\ \lambda_1 & \kappa_1 \end{vmatrix}}, \quad \underset{23}{+\begin{vmatrix} m_1 & n_1 \\ \mu_1 & \nu_1 \end{vmatrix}}, \quad \underset{24}{-\begin{vmatrix} m_1 & k_1 \\ \nu_1 & \kappa_1 \end{vmatrix}}, \quad \underset{34}{+\begin{vmatrix} n_1 & k_1 \\ \nu_1 & \kappa_1 \end{vmatrix}};$$

$\varepsilon_{ijkl}\, a^k b^l$ $\qquad \varepsilon_{\boxed{34}12}$

ε_{2413}

2413 #

1243

2314 #

1324 |

14.23 #

$$\mathfrak{M}_{34} = \begin{vmatrix} l_1 & m_1 \\ \lambda_1 & \mu_1 \end{vmatrix}$$

$$\mathfrak{M}_{24} = -\begin{vmatrix} l_1 & n_1 \\ \lambda_1 & \nu_1 \end{vmatrix}$$

$$\mathfrak{M}_{23} = +\begin{vmatrix} l_1 & k_1 \\ \lambda_1 & \kappa_1 \end{vmatrix}$$

$$\mathfrak{M}_{12} = +\begin{vmatrix} n_1 & k_1 \\ \nu_1 & \kappa_1 \end{vmatrix}$$

$$\mathfrak{M}_{13} = -\begin{vmatrix} m_1 & k_1 \\ \nu_1 & \kappa_1 \end{vmatrix}$$

$$\mathfrak{M}_{14} = +\begin{vmatrix} m_1 & n_1 \\ \mu_1 & \nu_1 \end{vmatrix}$$

$$\left\| \begin{matrix} x^1 - x^1_0 & x^2 - x^2_0 & x^3 - x^3_0 & x^4 - x^4_0 \\ a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \end{matrix} \right\| = 0 \quad \text{plano } \Pi$$

$$\mathfrak{M}_{ij} = \varepsilon_{ijkl}\, a^k b^l$$

$$\mathfrak{N}_{ij} = \varepsilon_{ijpq}\, c^p d^q$$

$$\begin{vmatrix} n_1 & k_1 \\ \nu_1 & \kappa_1 \end{vmatrix};$$

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Transformación de coordenadas:

$$\bar{a}^k = \lambda^k_r\, a^r \qquad \bar{b}^l = \lambda^l_s\, b^s$$

$$\overline{\mathfrak{M}}_{ij} = \varepsilon_{ijkl}\, \lambda^k_r\, \lambda^l_s\, a^r b^s$$

$$\overline{\mathfrak{N}}_{ij} = \varepsilon_{ijpq}\, \lambda^p_t\, \lambda^q_v\, c^t d^v$$

$\bar{A}_1 = a_1 A + a_2$

$\bar{B}_1 =$

$\bar{C}_1 =$

$\bar{D}_1 =$

$\bar{A}^i = a^i_j A^j$ rotación

$\bar{B}^i = a^i_j B^j$

$$\begin{vmatrix} A^i & A^j \\ B^i & B^j \end{vmatrix} = \begin{vmatrix} a^i_p A^p & a^j_q A^q \\ a^i_r B^r & a^j_s B^s \end{vmatrix} =$$

$$\begin{vmatrix} a^i_1 A^1 + a^i_2 A^2 + a^i_3 A^3 + a^i_4 A^4 & a^j_1 A^1 + a^j_2 A^2 + a^j_3 A^3 + a^j_4 A^4 \\ a^i_1 B^1 + a^i_2 B^2 + a^i_3 B^3 + a^i_4 B^4 & a^j_1 B^1 + a^j_2 B^2 + a^j_3 B^3 + a^j_4 B^4 \end{vmatrix}$$

$$= \begin{vmatrix} a^i_1 & a^j_1 \\ a^i_1 & a^j_1 \end{vmatrix} \cdot \begin{vmatrix} a^j_1 A^1 & a^j_2 A^2 \\ a^i_1 B^1 & a^j_2 B^2 \end{vmatrix}$$

$$\begin{vmatrix} a^i_1 A^1 + a^i_2 A^2 & a^j_1 A^1 + a^j_2 A^2 \\ a^i_1 B^1 + a^i_2 B^2 & a^j_1 B^1 + a^j_2 B^2 \end{vmatrix} = \begin{vmatrix} a^i_1 & a^i_2 \\ & \end{vmatrix} \begin{vmatrix} A^1 \\ \end{vmatrix}$$

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$$\begin{vmatrix} a_1^i A^1 & a_1^j A^1 \\ a_1^i B^1 & a_1^j B^1 \end{vmatrix} = 0$$

$$A^1 B^2 (a_1^i a_2^j - a_1^j a_2^i) - A^2 B^1 (a_1^i a_2^j - a_2^i a_1^j)$$

$$= \begin{vmatrix} A^1 & A^2 \\ B^1 & B^2 \end{vmatrix} \begin{vmatrix} a_1^i & a_2^i \\ a_1^j & a_2^j \end{vmatrix}$$

$$\boxed{\mathfrak{a}_{pq}^{ij} \quad \mathfrak{M}^{pq}}$$

$$\mathfrak{M}^{pq} = \begin{vmatrix} A^p & A^q \\ B^p & B^q \end{vmatrix} \qquad \mathfrak{a}_{pq}^{ij} = \begin{vmatrix} a_p^i & a_p^j \\ a_q^i & a_q^j \end{vmatrix}$$

$$\cancel{\overline{\mathfrak{M}}^{pq} = \mathfrak{a}}$$

$$\boxed{\overline{\mathfrak{M}}^{ij} = \mathfrak{a}_{pq}^{ij} \, \mathfrak{M}^{pq}}$$

$$\begin{vmatrix} \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 \end{vmatrix} \cdot 1 \end{vmatrix}$$

$$\left\| \begin{matrix} \bar{x} - x & \bar{y} - y & \bar{z} - z & \bar{v} - v \\ l & m & n & k \\ \lambda & \mu & \nu & \kappa \end{matrix} \right\| = 0$$

cuadrivector unitario

cuadrivector perpendicular al primero y también unitario.

$$l\lambda + m\mu + n\nu + k\kappa = 0$$

$$\lambda^2 + m^2 + n^2 + k^2 = 1$$

$$\begin{vmatrix} l & m \\ \lambda & \mu \end{vmatrix}^2 + \begin{vmatrix} l & n \\ \lambda & \nu \end{vmatrix}^2 + \begin{vmatrix} l & k \\ \lambda & \kappa \end{vmatrix}^2 + \begin{vmatrix} m & n \\ \mu & \nu \end{vmatrix}^2 + \begin{vmatrix} m & k \\ \mu & \kappa \end{vmatrix}^2 + \begin{vmatrix} n & k \\ \nu & \kappa \end{vmatrix}^2 = ?$$

$$(l\mu - m\lambda)^2 + (l\nu - n\lambda)^2 + (l\kappa - k\lambda)^2 + (m\nu - n\mu)^2 + (m\kappa - k\mu)^2 + (n\kappa - k\mu)^2 +$$

$$\begin{aligned} & l^2(\mu^2 + \nu^2 + \kappa^2 + \lambda^2) && - l^2\lambda^2 \\ + & m^2(\lambda^2 + \nu^2 + \kappa^2 + \mu^2) && - m^2\mu^2 \\ + & n^2(\lambda^2 + \mu^2 + \kappa^2 + \nu^2) && - n^2\nu^2 \\ + & k^2(\lambda^2 + \mu^2 + \nu^2 + \kappa^2) && - k^2\kappa^2 \end{aligned}$$

$$-2lm\mu\lambda - 2ln\nu\lambda - 2lk\lambda\kappa - 2mn\mu\nu - 2mk\mu\kappa - 2nk\nu$$

$$= 1 - (l\lambda + m\mu + n\nu + k\kappa)^2 = 1$$

Sextivector normalizado.

Cuadrivector y Cuadrivector $\perp$

~~Cuadrado de la m~~

Magnitud del sextivector es igual al producto de las magnitudes de los cuadrivectores.

ector unitario
ector perpendicular
nero y también
itario.

$k|^2 = ?$
$k|$

$+ \quad (nk - k\mu)^2 + (nk - k\nu)^2 = ?$

^{2}

$,^2$

κ^2

$mk \quad \mu k - 2nk\nu k$

$$\Pi_1 : \left\| \begin{matrix} \bar{x}-x & \bar{y}-y & \cdots & \cdots & \cdots \\ x' & y'' & & & \\ x'+x''s+\frac{1}{2}x'''s^2 & y'+y''s+\frac{1}{2}y'''s^2 & & & \end{matrix} \right\| = 0$$

$$\Pi_2 : \left\| \begin{matrix} \bar{x}-x-x's-\frac{1}{2}x''s^2 & \bar{y}-y-y's-\frac{1}{2}y''s^2 & \cdots \\ x'+x''s+\frac{1}{2}x'''s^2 & y'+y''s+\frac{1}{2}y'''s^2 & \cdots \\ x'+x''\sigma+\frac{1}{2}x'''\sigma^2 & y'+y''\sigma+\frac{1}{2}y'''\sigma^2 & \cdots \end{matrix} \right\| = 0$$

$$\left[(x'y''-x''y')s+\tfrac{1}{2}(x'y'''-x'''y')s^2\right], \ldots$$

(Norma)^2 del sextivector corresp. a Π_1:

$$N_1^2 = \begin{cases} x'^2\left[y''^2+z''^2+v''^2+x''^2\right]s^2 - x'^2x''^2s^2 \\ +y'^2\left[x''^2+y''^2+z''^2+v''^2\right]s^2 - y'^2y''^2s^2 \\ -2x'x''y'y''-2x'x''z'z'' - \cdots \end{cases}$$

$$N_1^2 \simeq s_{22}S^2$$

$$N_1 \approx \sqrt{s_{22}}\, S \qquad N_1 \approx \frac{s}{\rho}$$

(Sextivector)$_1$ normalizado:

$$(x'y''-x''y')\rho+\tfrac{1}{2}(x'y'''-x'''y')\rho s, \ldots$$

$$\boxed{\begin{array}{l}\text{Sextivector normalizado del plano osculador:}\\ (x'y''-x''y')\rho\ ,\ \cdots\end{array}}$$

Sextivector del plano ~~vecino~~ osculador vecino.

$\cancel{(x'y'''-x'''y')\rho+}$

$$(x'y''-x''y')\rho+(x'y'''-x'''y')\rho s+(x'y''-x''y')\rho' s,\ \dots$$

$$(\text{Norma}_2)^2=1+2\sum(x'y''-x''y')(x'y'''-x'''y')\rho^2 s$$
$$+2\frac{\rho'}{\rho}s+\cdots\cdots$$

$$x'^2(y''y'''+z''z'''+v''v''')\rho^2 s-x'^2x''x'''\rho^2 s$$
$$+y'^2(\qquad\qquad)\rho^2 s-\cancel{y'^2y''y'''\rho^2 s}$$

$$-x'x'''y'y''-x'x''y'y'''$$

$$=S_{23}\rho^2 s-(x'x''+y'y''+\cdots)(x'x'''+y'y'''+\cdots)\rho^2 s$$

$\cancel{=S_{23}\rho^2}$

$$=-\frac{\rho'}{\rho}s+0$$

$$(\text{Norma}_2)^2\approx 1$$

$$\cos \theta = 1 - \frac{\rho'}{\rho} s + \frac{\rho'}{\rho} s$$

$$\mathfrak{M}_1 \quad \mathfrak{M}_2 \; - - \; \mathfrak{M}_6$$

$$\mathfrak{N}_1 \quad \mathfrak{N}_2 \; \text{———} \; \mathfrak{N}_6$$

$$(\mathfrak{M}_1 \mathfrak{N}_2 - \mathfrak{M}_2 \mathfrak{N}_1)^2 =$$

$$\begin{array}{l|l} \mathfrak{M}_1^2 (\mathfrak{N}_2^2 + \mathfrak{N}_3^2 + \mathfrak{N}_4^2 + \mathfrak{N}_1^2) & - \mathfrak{M}_1 \mathfrak{N}_1 \\ + \mathfrak{M}_2^2 (\mathfrak{N}_1^2 \; \mathfrak{N}_2^2 - -) & - \mathfrak{M}_2 \mathfrak{N}_2 \end{array}$$

$$- \mathfrak{M}_1 \mathfrak{M}_2 \mathfrak{N}_1 \mathfrak{N}_2 - - -$$

$$(x' y'' - x'' y') \rho, \quad (x' z'' - x'' z') \rho, \quad \ldots$$

$\cancel{(x' y'' - x'' y')}$

$$(x'y'' - x''y')\rho$$

$$(x'y'' - x''y')\rho + \left\{\begin{matrix}(x'y''' - x'''y')\rho \\ +(x'y'' - x''y')\rho'\end{matrix}\right\} s$$

$$+ \frac{1}{2}\left[\begin{matrix}(x'y^{IV} - x^{IV}y')\rho + (x''y''' - x'''y'')\rho \\ +2(x'y''' - x'''y')\rho' + (x'y'' - x''y')\rho''\end{matrix}\right] s^2$$

$$(x'z'' - x''z')\rho$$

$$(x'z'' - x''z')\rho + \left\{\begin{matrix}(x'z''' - x'''z')\rho \\ +(x'z'' - x''z')\rho'\end{matrix}\right\} s$$

$$+ \frac{1}{2}\left[\begin{matrix}(x'z^{IV} - x^{IV}z')\rho + (x''z''' - x'''z'')\rho \\ +2(x'z''' - x'''z')\rho' + (x'z'' - x''z')\rho''\end{matrix}\right] s^2$$

$$\left\{(x'y'' - x''y')(x'z''' - x'''z') - (x'z'' - x''z')(x'y''' - x'''y')\right\}\rho^2$$